Volatility Regressions with Fat Tails*

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Abstract

Nowadays, a common practice to forecast integrated variance is to do simple OLS autoregressions of the observed realized variance data. However, non-parametric estimates of the tail index of this realized variance process reveal that its second moment is possibly unbounded. In this case, the behavior of the OLS estimators and the corresponding statistics are unclear. We prove that when the second moment of the spot variance is unbounded, the slope of the spot variance's autoregression converges to a random variable when the sample size diverges. Likewise, the same result holds when one consider either integrated variance's autoregression or the realized variance one. We then consider a class of variance models based on diffusion processes having an affine form of drift, where the class includes GARCH and CEV processes, and we prove that IV estimations with adequate instruments provide consistent estimators of the drift parameters as long as the variance process has a finite first moment regardless of the existence of finite second moment. In particular, for the GARCH diffusion model with fat tails, an IV estimation where the instrument equals the sign of the (demeaned) lagged value of the variable of interest provides consistent estimators. Simulation results corroborate the theoretical findings of the paper.

Key words: volatility; autoregression; fat tails; random limits.

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1. Introduction

In this paper, we are interested in using high-frequency based measures to forecast future variance. A common practice is to approximate the latent daily integrated variance by high-frequency based realized measures like realized variance (Andersen et al. (2001)) or robust-to-noise measures (Zhang et al. (2005); Barndorff-Nielsen et al. (2008); Jacod et al. (2009)), and then to estimate by OLS a simple autoregressive regression of this realized measures to get a forecast of the integrated variance. This autoregressive regression is often misspecified because the dynamics of the integrated variance is more complex than a simple autoregressive process. For instance, if the true instantaneous (or spot) variance is a square-root process, then the integrated and realized variances are ARMA (1,1) processes (Barndorff-Nielsen and Shephard (2002); Meddahi (2003)). Still, even if the autoregression model is misspecified, it provides a very accurate forecast because integrated variance as well as high-frequency realized measures are persistent and therefore few lags are sufficient to predict well future volatility (Andersen et al. (2003); Andersen et al. (2004)).

On the other hand, the GARCH era (Engle (1982); Bollerslev (1986)) based on parametric models of daily data provides very useful information about the variance process. One of them which is a primary interest in this paper is fat tails. When one estimates a daily GARCH model on stock returns or exchange rates, one often finds that the returns' fourth moment is not bounded or close to be unbounded. If the fourth moment of the returns is unbound, then the second moment of the daily realize variance defined as the sum of intra-daily squared returns is also unbounded. Consequently, the interpretation, based on L^2 projections, of the autoregressive regression and the OLS estimation are questionable. Likewise, the delivered forecast and all the statistical tools, relying on Gaussian limit theory, used to assess the quality of the forecast could be invalid.

The doubt about the finiteness of the fourth moment of the returns is based on a parametric model of the volatility. In contrast, an important contribution of the high-frequency volatility literature is that the availability of a lot of information allows one to get non-parametric measures of the variance, and therefore, to get rid of these volatility parametric models. It is therefore necessary to assess the finiteness of the second moment of realized measures in a non-parametric way. The solution hinges on a non-parametric estimation of the tail index. We use the Hill's (1975) estimator to our data and we get the same result. More precisely, we find that the Hill estimator of the tail index of the daily returns is close to four, while it is close to two for the daily realized variance, and two other popular measures which are robust to the presence of jumps, namely the bipower variance of Barndorff-Nielsen and Shephard (2004b, 2006) and the threshold variance estimator of Jacod (2008, 2012) and Mancini (2009).

In this paper, we revisit the results about the autoregressive regression of the variance process like Andersen et al. (2004) when the second moment of the spot variance is possibly unbounded, implying that the second moment of integrated and realized variances are unbounded.[1] When the instantaneous variance has an unbounded second moment, then the results in Andersen et al. (2004) are no more valid because one can not compute population autoregression parameters.

We study empirical regressions instead of population regressions. More precisely, we analyze the asymptotic behavior of the OLS estimator of the autoregressions. We consider autoregressions of three variables: the spot variance, the integrated variance and the realized variance. Of course, the first two autoregressions are not doable in practice because the variables are not observed, but still the two autoregressions provide good benchmarks. In particular, the third autoregression will try to mimic the second one.

The asymptotic behavior of an OLS estimator under fat tails is ambiguous and depends on the model. For instance, when one considers an autoregressive process of order one with i.i.d. errors and unbounded variance, the OLS estimator is consistent whether the autoregressive parameter is smaller than one (Hannan and Kanter (1977) and Knight (1987)) or equals to one (Chan and Tran (1989) and Phillips (1990)). In contrast, when one considers an ARCH or GARCH process, the autocorrelation parameters of the squared process converge to random variables when the fourth moment of the process is unbounded (Davis and Mikosch (1998) and Mikosch and Starica (2000)). It is therefore needed to study the behavior of the OLS estimator when one does an autoregression of volatility measures.

When we study the autoregressions, we consider two types of asymptotic ap-

[1] When one considers a continuous time model without jumps and without market microstructure noise, the fourth moment of the intra-day returns is unbounded if and only if the second moment of the instantaneous variance is infinite.

proaches. Two time-dimension variables will play a role in these asymptotic analysis: Δ which is the length of sub-periods and the time span denoted T. In the first asymptotic approach, we assume that $\Delta \to 0$ while T is fixed or diverges to ∞. We do this type of asymptotic because we want to characterize the behavior of the OLS estimators without making a parametric model assumption as did Andersen et al. (2004). In the second asymptotic approach, we keep Δ fixed and allow $T \to \infty$ at the cost of making a parametric assumption of the variance diffusion process.

In the first asymptotic analysis with $\Delta \to 0$, we characterize the behavior of the OLS estimators of the three regressions' slopes. When the spot variance process has a bounded second moment, we prove that the OLS estimators converge to finite quantities, which are the same ones as the population parameters derived in Andersen et al. (2004). In contrast, when the spot variance has an unbounded second moment, we prove that the OLS estimators converge to random variables. Both the simulations and the comparison with the results in Andersen et al. (2004) when the spot variance has a finite second moment corroborate the good quality of our approach.

These results are obviously negative. Providing positive results in a general context is not easy because one needs to specify the object of interest. We therefore consider a class of variance models based on diffusion processes having an affine form of drift, where the class includes GARCH and CEV processes, with possibly unbounded second moment. For this semiparametric class of models, we follow the literature on regressions with fat tails like Blattberg and Sargent (1971) and Samorodnitsky et al. (2007) by considering instrumental variable (IV) estimations. We prove that the IV estimators become consistent estimators of the drift parameters when instruments are chosen appropriately.

Samorodnitsky et al. (2007) studied the estimation of linear regression models where the explanatory and the noise variables have fat tails. It considered estimators that have an instrumental variable interpretation where the instrument is a signed power of the explanatory variable, with the OLS being a particular case. The choice of the power is selected for either efficiency purposes or for getting an estimator with a normal asymptotic distribution, which is often not the case of the OLS estimator when it is consistent. However, in this paper, we select the instruments for consistency purposes of the drift parameters. The asymptotic distribution of the estimator as well as the efficiency question are not studied and left for future research.

When Δ is fixed, unlike the asymptotics with $\Delta \to 0$, we need a conditional moment restriction for the asymptotics of IV estimators. It is well known for a stationary diffusion with affine drift that the conditional mean is also affine as long as the diffusion has a bounded second moment (see for instance Meddahi and Renault (2004)). We prove for a GARCH diffusion that the result is still valid when the second moment is unbounded. We then show that the IV estimation with adequate instruments leads to consistent estimators of the drift parameters. A particular instrument we study is the sign of the lagged value of the demeaned spot variance, corresponding to a power zero of the signed power instrument mentioned above. This estimator is first proposed by Cauchy (1836), and is often referred to as the "Cauchy estimator" (see, e.g., So and Shin (1999); Phillips et al. (2004); Choi et al. (2016) for the recent use of the Cauchy estimator).

Interestingly, Jean-Marie Dufour used in several studies sign-based methods for inference purposes, especially for exact inference in finite sample. In particular, he used such approach in Coudin and Dufour (2009, 2017) in order to provide inference about the slope parameter in a linear regression model without making moment restrictions on the disturbance errors and therefore allowing for fat tails. The assumption made in these papers is a median restriction on the errors conditional on the explanatory variables. In other words, we are using the same approach with a slightly different framework because we assume that the (conditional) first moment of the errors exists and equals zero but we do not make assumptions on higher moments.

The paper is organized as follows. The next section provides the setup, an empirical motivation for fat tails, and various regressions. In Section 3, we analyze the asymptotic behavior of the OLS estimators when $\Delta \to 0$. Section 4 studies the IV estimation, while Section 5 provides two extensions, including the estimation with a fixed Δ. Section 6 provides simulations to assess the finite sample properties of the estimators, while the last section concludes. All the proofs are provided in Appendix.

Throughout the paper we use "$P_T \sim Q_T$" to denote $P_T = Q_T(1+o(1))$. Similarly, "$P_T \sim_p Q_T$" and "$P_T \sim_d Q_T$" mean $P_T = Q_T(1+o_p(1))$ and $P_T =_d Q_T(1+o_p(1))$, respectively. These notations, as well as other standard notations used in asymptotics, will be used frequently throughout the paper without further references.

2. Model and Preliminaries

2.1. Spot, Integrated and Realized Variances

We consider a price process $(P_t, 0 \leq t \leq T)$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Our basic assumption is that P_t is a Brownian semimartingale with the following form:

$$d\log(P_t) = D_t dt + V_t^{1/2} dW_t^P,$$

where W_t^P is a Brownian motion, D_t and V_t are adapted processes with càdlàg paths. For a Δ-interval, we define the spot variance (v_i), integrated variance (x_i) and realized variance (y_i) of the price process (P_t) as

$$v_i = V_{i\Delta}, \qquad x_i = \frac{1}{\Delta}\int_{(i-1)\Delta}^{i\Delta} V_t dt, \qquad y_i = \frac{1}{\Delta}\sum_{j=1}^{n}\left(r^{(\delta)}_{(i-1)\Delta+j\delta}\right)^2, \tag{2.1}$$

for $i = 1, \cdots, N$ with $N\Delta = T$, where $r^{(\delta)}$ is the δ-period return defined as $r^{(\delta)}_{(i-1)\Delta+j\delta} = \log(P_{(i-1)\Delta+j\delta}) - \log(P_{(i-1)\Delta+(j-1)\delta})$ for $j = 1, \cdots, n$ with $n\delta = \Delta$. It is well known that the realized variance y is a noisy measure of the integrated variance x, and satisfies

$$(n/2)^{1/2}(y_i - x_i) \to_d \eta_i \mathbb{N}(0,1), \tag{2.2}$$

where $\eta_i^2 = \Delta^{-1}\int_{(i-1)\Delta}^{i\Delta} V_t^2 dt$, as $n \to \infty$ for fixed Δ and for each $i = 1, \cdots, N$. See, e.g., Barndorff-Nielsen and Shephard (2004a). Moreover, the convergence (2.2) holds jointly for $i = 1, \cdots, N$ if $T = N\Delta$ is fixed (see, e.g., Jacod and Protter (1998)).

In this paper, we analyze the asymptotic properties of various estimators for the volatility regression. Specifically, we consider the first order autoregression

$$z_{i+1} = \alpha_z + \beta_z^{(k)} z_{i-k} + u_{i+1} \quad \text{with} \quad k \geq 0 \tag{2.3}$$

for $z = v, x, y$, and estimate the slope coefficient $\beta_z^{(k)}$ using OLS or IV method. Our asymptotics for $z = v, x$ involve two parameters, the sampling interval Δ and the time span T, and it is developed under the assumption that $\Delta \to 0$ and $T \to \infty$ simultaneously. On the other hand, the asymptotics for $z = y$ involve three parameters, the sampling interval Δ at low-frequency, the sampling interval δ at high-frequency, and the time span T. In this case, the asymptotics are developed under the assumption that $\delta/\Delta \to 0$, $\Delta \to 0$ and $T \to \infty$ simultaneously.

To effectively analyze the large T asymptotics, we assume that the underlying variance process V is a diffusion process on $\mathcal{D} = (\underline{v}, \overline{v}) \subset \mathbb{R}$ driven by

$$dV_t = \mu(V_t)dt + \sigma(V_t)dW_t, \tag{2.4}$$

where W is a Brownian motion, and μ and σ are respectively drift and diffusion functions of V. To obtain more explicit asymptotic results, we mainly consider a pure diffusion V without having leverage effects, i.e., each of V and D is independent of W^P, unless we mention that they are dependent. We believe that the implications of our results under no leverage effect remain valid for the model with leverage effects.

We let s be the scale function defined as

$$s(v) = \int_y^v \exp\left(-\int_y^x \frac{2\mu(z)}{\sigma^2(z)}dz\right)dx, \tag{2.5}$$

where the lower limits of the integrals can be arbitrarily chosen to be any point $y \in \mathcal{D}$. Defined as such, the scale function s is uniquely identified up to any increasing affine transformation, i.e., if s is a scale function, then so is $as + b$ for any constants $a > 0$ and $-\infty < b < \infty$. We also define the speed density

$$m(v) = \frac{1}{(\sigma^2 s')(v)} \tag{2.6}$$

on $\mathcal{D}$, where s' is the derivative of s, often called the scale density, which is assumed to exist. The speed density is defined to be the measure on $\mathcal{D}$ given by the speed density with respect to the Lebesgue measure.

Throughout this paper, we assume

Assumption 2.1. *(a) $\sigma^2(v) > 0$ for all $v \in \mathcal{D}$, and (b) $\mu(v)/\sigma^2(v)$ and $1/\sigma^2(v)$ are locally integrable at every $v \in \mathcal{D}$.*

Assumption 2.1 provides a simple sufficient set of conditions to ensure that a weak solution to the stochastic differential equation (2.4) exists uniquely up to an explosion time. See, e.g., Theorem 5.5.15 in Karatzas and Shreve (1991). Note, under Assumption 2.1, that both the scale function s and speed density m are well defined, and that the scale function is strictly increasing, on $\mathcal{D}$. Consequently, the natural scale diffusion V^s of V, where $V^s = s(V)$, is well defined with speed density $m_s = (m/s') \circ s^{-1}$. It follows immedaitely from Ito's lemma that the natural scale diffusion V^s has no drift term. Following Kim and Park (2017), we use the natural scale representation in the development of our long span asymptotics.

2.2. *Population Regressions with GARCH Diffusions*

In this section, we study the volatility regressions in population. Under $\mathbb{E}(V_t^2) < \infty$, Andersen et al. (2004) analyzed the volatility regressions in population. These authors considered the Eigenfunction Stochastic Volatility (ESV) model of Meddahi (2001) to derive analytical forecast results. Examples of ESV includes the square-root model, the log-normal stochastic volatility model and the GARCH diffusion model. We focus here on the GARCH diffusion model of Nelson (1990) because this example allows for unbounded moments while the two other ones lead to bounded ones. More precisely, assume that the spot variance process V_t, defined on $(0, \infty)$, is given by

$$dV_t = \kappa(\mu - V_t)dt + \sigma V_t dW_t. \tag{2.7}$$

Under the stationarity conditions on V_t, one can easily prove that the second moment of V_t is bounded if and only if $\sigma^2 < 2\kappa$.

2.2.1 GARCH Diffusions with $\mathbb{E}(V_t^2) < \infty$

Andersen et al. (2004) computed the population values of the autocovariances of spot (v), integrated (x) and realized variances (y) under $\mathbb{E}(V_t^2) < \infty$. From these quantities, one gets the corresponding autoregressive coefficients β_v, β_x and β_y. In particular, one has

$$\beta_v = \exp(-\kappa\Delta), \quad \beta_x = \frac{1}{2}\frac{(1-\exp(-\kappa\Delta))^2}{\exp(-\kappa\Delta) + \kappa\Delta - 1}, \quad \beta_y = \frac{a_1^2}{\Delta^2\kappa^2}\frac{(1-\exp(-\kappa\Delta))^2}{Var(y)},$$

where

$$Var(y) = 2\frac{a_1^2}{\Delta^2\kappa^2}(\exp(-\kappa\Delta) + \kappa\Delta - 1) + \frac{4}{\delta\Delta}\left(\frac{a_0^2\delta^2}{2} + \frac{a_1^2}{\kappa^2}(\exp(-\kappa\delta) + \kappa\delta - 1)\right),$$

with $a_0 = \mathbb{E}(V_t) = \mu$ and $a_1^2 = Var(V_t) = \mu^2\sigma^2/(2\kappa - \sigma^2)$.

One should notice that in this example, the spot variance is an AR(1) process while both integrated and realized variances are ARMA(1,1) processes. In addition, the three processes have the same autoregressive root which equals $\exp(-\kappa\Delta)$.

When Δ is small, one gets

$$\beta_v = 1 - \kappa\Delta + o(\Delta), \qquad \beta_x = 1 - \frac{2}{3}\kappa\Delta + o(\Delta).$$

Likewise, when both Δ and δ/Δ are small, one gets

$$\beta_y = 1 - \frac{2}{3}\kappa\Delta - 2\frac{\delta}{\Delta}\frac{\mathbb{E}(V_t^2)}{Var(V_t)} + o(\Delta) + o(\delta/\Delta)$$

with $Var(V_t) = \mu^2\sigma^2/(2\kappa - \sigma^2)$ and $\mathbb{E}(V_t^2) = 2\kappa\mu^2/(2\kappa - \sigma^2)$. It is interesting to notice that, as $\delta/\Delta, \Delta \to 0$, we have

$$\beta_v - 1 \sim -\Delta\kappa, \qquad \beta_x - 1 \sim -\Delta\frac{2}{3}\kappa, \qquad \beta_y - 1 \sim -\Delta\frac{2}{3}\kappa - 4\frac{\delta}{\Delta}\frac{\kappa}{\sigma^2}, \tag{2.8}$$

that is, integrated variance has a larger first order autocorrelation than the spot and realized variances.

2.2.2 GARCH Diffusions with $\mathbb{E}(V_t) < \infty$

One can easily prove that

$$V_{t+\Delta} = \mu + \exp(-\kappa\Delta)(V_t - \mu) + \varepsilon_{t+\Delta}, \quad \varepsilon_{t+\Delta} = \sigma\int_t^{t+\Delta} \exp(-\kappa(t+\Delta-u))V_u du.$$

When $\mathbb{E}(V_t^2) < \infty$, $\varepsilon_{t+\Delta}$ is a martingale-difference-sequence (m.d.s.), which implies that

$$\mathbb{E}[V_{t+\Delta} \mid V_t] = \mu + \exp(-\kappa\Delta)(V_t - \mu). \tag{2.9}$$

However, the m.d.s. result is not valid when $\mathbb{E}(V_t^2) = \infty$ because $\int_0^{t+\Delta} \exp(-\kappa(t+\Delta-u))V_u du$ is not a martingale but a local martingale. Interestingly, we are able to prove that (2.9) is still valid when V_t is a stationary GARCH diffusion with $\mathbb{E}(V_t) < \infty$, whether $\mathbb{E}(V_t^2)$ is finite or not.[2]

Lemma 2.1. *For any* $\Delta > 0$*, we have*

$$\mathbb{E}[(v_{i+1} - \mu) - \exp(-\kappa\Delta)(v_i - \mu)|v_i] = 0. \tag{2.10}$$

When $\mathbb{E}(V_t^2) < \infty$, the previous result implies that V_t is an AR(1), from which one can estimate $\exp(-\kappa\Delta)$ by using an autoregression of order one of the spot variance. However, both the integrated and realized variances are ARMA(1,1) processes, which implies that first order autoregressions of these variables will not deliver a consistent estimator of the autoregressive parameter $\exp(-\kappa\Delta)$. However, Meddahi (2003) derived multiperiod moment restrictions fulfilled by the integrated and realized variances when $\mathbb{E}(V_t^2) < \infty$. The following result proves that these multiperiod moment restrictions are still valid when $\mathbb{E}(V_t^2) = \infty$.

Proposition 2.2. *Let* $\Delta > 0$*. (a) For* $z = v, x$*, we have*

$$\mathbb{E}[(z_{i+1} - \mu) - \exp(-\kappa\Delta)(z_i - \mu)|z_{i-1}] = 0. \tag{2.11}$$

(b) If $D_t = 0$ *almost surely for all* $t \geq 0$*, then the result in Part (a) holds for* $z = y$*.*

Proposition 2.2 will allow us to estimate consistently the coefficient $\exp(-\kappa\Delta)$ even when $\mathbb{E}(V_t^2) = \infty$ by using the following corollary:

Corollary 2.3. *Let* $r : \mathbb{R} \to \mathbb{R}$ *be bounded such that* $\mathbb{E}[(z_i - \mu)r(z_{i-1} - \mu)] \neq 0$ *for a given* $\Delta > 0$*. If* $D_t = 0$ *almost surely for all* $t \geq 0$*, we have*

$$\frac{\mathbb{E}[(z_{i+1} - \mu)r(z_{i-1} - \mu)]}{\mathbb{E}[(z_i - \mu)r(z_{i-1} - \mu)]} = \exp(-\kappa\Delta)$$

for $z = v, x, y$*.*

2.3. Empirical Evidences of Fat Tails

We now assess the magnitude of tails of empirical data. We use trade data on the SPDR S&P 500 ETF (SPY), which is an exchange traded fund (ETF) that tracks

[2] We are very grateful to Jean Jacod for providing us the proof of the result.

the S&P 500 index. Our primary sample comprises 10 years of trade data on SPY starting from June 15, 2004 through June 13, 2014 as available in the New York Stock Exchange Trade and Quote (TAQ) database. This tick-by-tick dataset has been cleaned according to the procedure outlined by Barndorff-Nielsen et al. (2008). We also remove short trading days leaving us with 2,497 days of trade data. In addition, we consider three subperiods: Before Crisis, from June 15, 2004 through August 29, 2008 (1,053 trading days); During Crisis, from September 2, 2008 through May 29, 2009 (185 trading days), and After Crisis, from June 1, 2009 through June 13, 2014 (1,259 trading days).

We estimate the tail index of the daily open-to-close returns and daily realized variance based on five minutes intra-day returns. Because we could have jumps that may affect the tail of the realized variance data, we also consider daily bipower variation which is a consistent estimator of integrated variance under the presence of jumps (see Barndorff-Nielsen and Shephard (2003, 2004b, 2006); Barndorff-Nielsen et al. (2005); Barndorff-Nielsen et al. (2006)) as well as the threshold estimator of integrated variance (see Jacod (2008, 2012); Mancini (2009); Jacod and Rosenbaum (2013)).

We estimate the tail index by using the Hill's (1975) estimator. Let $(X_i)_{i=1}^n$ be a stationary time series with

$$\mathbb{P}[X_i > x] \sim x^{-\alpha}\ell(x), \quad x \to \infty,$$

for some slowly varying function ℓ. The Hill's estimator for α^{-1} which arose in the i.i.d. context as a conditional MLE is defined as

$$h = \frac{1}{k_n}\sum_{i=1}^{k_n} \log(X_{(i)}/X_{(k_n)}),$$

where $(X_{(i)})_{i=1}^n$ is the order statistics $X_{(n)} \leq \cdots \leq X_{(k_n)} \leq \cdots \leq X_{(1)}$ for some $k_n \leq n$ such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$.

The results by Hsing (1991) and Resnick and Stărică (1995) indicate that the Hill estimator is asymptotically quite robust with respect to deviations from independence; Resnick and Stărică (1998) prove consistency under ARCH-type dependence. See also Hill (2010) for some other processes including ARFIMA, FIGARCH, explosive

GARCH, nonlinear ARMA-GARCH and etc.

Valid standard errors of the Hill estimator are available for some specific models with serial correlation. Therefore, we will not provide any of them. Instead, we follow the literature by providing Hill's plots, that is by varying the integer k_n. A flat area is viewed as a good estimator of the tail index. As usual, we truncate k_n. In practice we start with $k = 25$.

Figure 1 depicts the Hill index of the returns and three volatility measures over the whole period for k_n between 25 and 500. The first panel provides the estimator for the returns, which is clearly below four. The second panel depicts the tail index of the three volatility measures. The plots suggest that the tail of these measures is below two. Observe that the three plots have flat areas, with a tail index between 1.2 and 1.4. One should notice that the plots for the three volatility measures are quite close.

The period considered in the previous figure includes the financial crisis. A natural question is whether the strong empirical evidence of fat tails is driven by the crisis' period. We therefore carry the Hill estimators for the periods before, during, and after crisis, as explained above. Given the length of the crisis period (185 trading days), we vary k_n from 25 to 150. Figure 2 depicts the Hill index of the returns on the top panel and the realized volatility on the bottom panel for the three periods while Figure 3 depicts those of the bipower (top panel) and threshold (bottom panel) measures. Clearly, the crisis period exhibits fatter tails than the other two periods for the four variables. However, both the periods before and after the crisis suggest very fat tails with a tail index slightly below four for the returns and around two for the three volatility measures. Therefore, the evidence of fat tails and unbounded second moment for the volatility measures is quite strong.

3. Least Square Estimates

In this section, we consider the OLS estimator $\hat{\beta}_z^{(k)}$ for $\beta_z^{(k)}$ in (2.3) given by

$$\hat{\beta}_z^{(k)} = \frac{\sum_{i=k+1}^{N-1}(z_{i-k} - \overline{z}_N)z_{i+1}}{\sum_{i=k+1}^{N-1}(z_{i-k} - \overline{z}_N)^2}$$

where $\overline{z}_N$ is the sample mean of $(z_{i-k} : i = k+1, \cdots, N-1)$. For $k = 0$, we simply write $\beta_z^{(0)} = \beta_z$ and $\hat{\beta}_z^{(0)} = \hat{\beta}_z$.

3.1. *Primary Asymptotics*

Recall that $T = N\Delta$ and $\Delta = n\delta$. For our asymptotics here we let $\delta/\Delta, \Delta \to 0$, with T being fixed or $T \to \infty$ simultaneously as $\delta/\Delta, \Delta \to 0$. In case we have $\delta/\Delta, \Delta \to 0$ and $T \to \infty$ simultaneously, we assume that $\delta/\Delta, \Delta \to 0$ sufficiently fast relative to $T \to \infty$. It is indeed more relevant in a majority of practical applications, which rely on observations collected at small sampling intervals over moderately long span.

In our asymptotics, we frequently deal with various functional transforms of D and V over time interval $[0, T]$. To effectively handle such functional transforms, we define

$$T_D = \max_{0 \le t \le T} |D_t| \quad \text{and} \quad T(f) = \max_{0 \le t \le T} |f(V_t)|$$

for some function $f : \mathcal{D} \to \mathbb{R}$. We also denote by ι the identity function on $\mathcal{D}$, and $\iota(v) = v$ for all $v \in \mathcal{D}$. Consequently, we have $T(\iota) = \max_{0 \le t \le T} |V_t|$ for the identity function. Obviously, T_D and $T(\iota)$ are the asymptotic orders of extremal process of D and V, respectively. The order of the extremal process is known for a wide class of diffusions. For instance, under some regularity conditions, the extremal process of a stationary diffusion V is of order $O_p(s^{-1}(T))$, where s is the scale function of V, to which the reader is referred to, e.g., Davis (1982). More generally, we may obtain the exact rate of $T(f)$ from the asymptotic behavior of extremal process. In particular, if f is regularly varying and c_T is the order of the extremal process, then the asymptotic order of $T(f)$ is given by $O_p(f(c_T))$.

Assumption 3.1. *(a) σ^2 is twice continuously differentiable on $\mathcal{D}$, and (b) for $f = \mu$, σ^2, $\sigma^{2\prime}$, $\sigma^{2\prime\prime}$ and ι, there is a locally bounded function $\omega : \mathcal{D} \to \mathbb{R}$ such that $|f(v)| \le \omega(v)$ for all $v \in \mathcal{D}$.*

The differentiability condition of σ^2 in Assumption 3.1 (a) is routinely assumed in the study of diffusion models. Under Assumption 3.1 (a), the majorizing function ω in Assumption 3.1 (b) always exists as long as μ is locally bounded.

Assumption 3.2. *For ω in Assumption 3.1, $\Delta T(\omega^8) T^2 \log(T/\Delta) \to_p 0$.*

Assumption 3.3. *For ω in Assumption 3.1,* $(\delta/\Delta)T(\omega^8)T^2\log^3(T/\delta) \to_p 0$.

Assumption 3.4. $(\delta/\Delta)T_D^4 T \to_p 0$.

Assumption 3.5. $(\delta/\Delta^2) = O(1)$.

Assumption 3.2 is similar to Assumption 5.1 in Kim and Park (2017), and provides a sufficient condition for our primary asymptotics of spot variance (v_i) and integrated variance (x_i). On the other hand, the asymptotics of realized variance (y_i) involve three parameters, δ, Δ and T, and require Assumptions 3.3-3.5 in addition to Assumption 3.2. The role of Assumption 3.3 is to analyze the asymptotic effect of the errors $(x_i - y_i)$ in the OLS estimates. On the other hand, Assumption 3.4 is a condition to control the effects from the drift part (D_t) in (P_t) so that (D_t) has no asymptotic impact in the asymptotics of the OLS estimates with (y_i). Lastly, Assumption 3.5 is to exclude less interesting cases where the errors $(x_i - y_i)$ dominate the signals (x_i) in the OLS estimates with (y_i). In particular, if $\delta/\Delta^2 \to \infty$, then the error components may have bigger stochastic order than the signals.

Assumptions 3.2-3.4 make it necessary to have $\Delta \to 0$ and $\delta/\Delta \to 0$. For a fixed T, a set of necessary and sufficient conditions for Assumptions 3.2-3.4 is $\Delta \to 0$ and $(\delta/\Delta)\log^3(1/\delta) \to 0$. Our asymptotics in the paper are derived under the conditions $\Delta \to 0$, $\delta/\Delta \to 0$ and $T \to \infty$ jointly. For Assumptions 3.2-3.4 to hold, it requires $\Delta \to 0$ and $\delta/\Delta \to 0$ sufficiently fastly as $T \to \infty$. For instance, Assumption 3.2 holds as long as $\Delta = O(T^{-2-\epsilon})$ for some $\epsilon > 0$, if V is bounded with $T(\omega^8) = O_p(1)$. For example, if daily observations over five years are available, then $\Delta = 1/250$ and $T^{-2} = 1/25$.[3] Our asymptotics in this section hold jointly in δ, Δ and T under Assumptions 3.1-3.5, and we do not use sequential asymptotics, requiring $\delta/\Delta \to 0$, $\Delta \to 0$ and $T \to \infty$ sequentially.

To effectively explain our asymptotics, we apply the summation by parts to the

[3]In our framework, the length Δ of day is a relative concept, and should be defined with the length of year simultaneously. If we set $\Delta = 1/250$, then $T = 1$ becomes a year. However, if we set $\Delta = 1$, then $T = 250$ becomes a year. Similarly, the sampling interval δ of the intraday observations should be defined with the length Δ of the day.

numerator of $\hat{\beta}_z^{(k)}$, and rewrite it as

$$\begin{aligned}\hat{\beta}_z^{(k)} - 1 =& \frac{1}{2}\frac{\sum_{j=0}^{k}\left((z_{N-j}^2 - z_{1+j}^2) - \overline{z}_N(z_{N-j} - z_{1+j})\right)}{\sum_{i=k+1}^{N-1}(z_{i-k} - \overline{z}_N)^2} \\ &- \frac{1}{2}\frac{\sum_{i=k+1}^{N-1}(z_{i+1} - z_{i-k})^2}{\sum_{i=k+1}^{N-1}(z_{i-k} - \overline{z}_N)^2}.\end{aligned} \tag{3.1}$$

For each term in (3.1), we have the following continuous time approximations when $\Delta \to 0$ and $\delta/\Delta \to 0$ such that Assumptions 3.1 and 3.2 holds.

Lemma 3.1. *Let Assumptions 3.1-3.5 hold.*

(a) For $k \geq 0$, we have

$$\sum_{j=0}^{k}\left((z_{N+j}^2 - z_{1+j}^2) - \overline{z}_N(z_{N+j} - z_{1+j})\right) \sim_p (1+k)\left(V_T^2 - V_0^2 - \overline{V}_T(V_T - V_0)\right),$$

$$\sum_{i=k+1}^{N-1}(z_{i-k} - \overline{z}_N)^2\Delta \sim_p \int_0^T (V_t - \overline{V}_T)^2 dt,$$

where $\overline{V}_T = T^{-1}\int_0^T V_t dt$, for $z = v, x, y$.

(b) For $k \geq 0$, we have

$$\sum_{i=k+1}^{N-1}(z_{i+1} - z_{i-k})^2 \sim_p \sum_{i=1}^{N-1}(z_{i+1} - z_i)^2 + k\sum_{i=1}^{N-1}(v_{i+1} - v_i)^2$$

for $z = v, x, y$, and

$$\sum_{i=1}^{N-1}(z_{i+1} - z_i)^2 \sim_p \begin{cases} [V]_T, & \textit{for } z = v \\ (2/3)[V]_T, & \textit{for } z = x \\ (2/3)[V]_T + (4\delta/\Delta^2)\int_0^T V_t^2 dt, & \textit{for } z = y. \end{cases}$$

Remark 3.1. (a) The continuous time approximations of the sum of squared increments (SSI), $\sum_{i=k+1}^{N-1}(z_{i+1} - z_i)^2$, in Lemma 3.1 (b) are depending upon z. In particular, we have

$$\sum_{i=k+1}^{N-1}(x_{i+1}-x_i)^2<\sum_{i=k+1}^{N-1}(v_{i+1}-v_i)^2, \tag{3.2}$$
$$\sum_{i=k+1}^{N-1}(x_{i+1}-x_i)^2<\sum_{i=k+1}^{N-1}(y_{i+1}-y_i)^2 \tag{3.3}$$

with probability approaching one as $\delta/\Delta,\Delta\to 0$ under Assumptions 3.1-3.5. An intuitive explanation for the inequalities in (3.2) and (3.3) are as follow. We can naturally expect that the integrated variance (x_i) has more smoother sample path compare to that of the spot variance (v_i). As a result, the SSI of (x_i) tends to be smaller than that of (v_i), and we have the first inequality in (3.2). On the other hand, the realized variance (y_i) is a noisy measure of the integrated variance (x_i), and the error component in (y_i) generates additional variation. Consequently, the sample path of (y_i) becomes more rougher compare to that of (x_i), and hence, (3.3) holds.

(b) Unlike Lemma 3.1 (b), the continuous time approximations in Lemma 3.1 are identical for all $z=v,x,y$. The results in Lemma 3.1 (a) are well expected since $|z_i-V_{(i-1)\Delta}|\to_p 0$ for all z as long as δ/Δ and Δ are sufficiently small relative to T.

(c) It follows from Ito's lemma and Lemma 3.1 with $k=0$ that

$$\begin{aligned}&\sum_{i=1}^{N-1}(z_i-\overline{z}_N)(z_{i+1}-z_i)\\ &\sim_p\begin{cases}\int_0^T(V_t-\overline{V}_T)dV_t, & \text{for } z=v\\ \int_0^T(V_t-\overline{V}_T)dV_t+(1/6)[V]_T, & \text{for } z=x\\ \int_0^T(V_t-\overline{V}_T)dV_t+(1/6)[V]_T-(2\delta/\Delta^2)\int_0^T V_t^2dt, & \text{for } z=y\end{cases}\end{aligned} \tag{3.4}$$

as $\delta/\Delta,\Delta\to 0$ under Assumptions 3.1-3.5. The result (3.4) for $z=v$ is quite natural and expectable by the asymptotic negligibility of discretization errors when $\Delta\to 0$. In a similar argument, one may expect

$$\sum_{i=1}^{N-1}(z_i-\overline{z}_N)(z_{i+1}-z_i)\sim_p\int_0^T(V_t-\overline{V}_T)dV_t \quad \text{for} \quad z=x,y \tag{3.5}$$

since $\sup_{0\leq i\leq N}|z_i - v_i| \to_p 0$ as $\delta/\Delta, \Delta \to 0$. However, we have (3.4), and the conjecture (3.5) is not true. This is not surprising at all since the convergence of stochastic process does not necessarily imply the convergence of stochastic integral associated with the stochastic process. The reader is referred to Kurtz and Protter (1991) for more detailed discussions about the weak convergence of stochastic integrals.

The primary asymptotics for $\hat{\beta}_z^{(k)}$ can be easily obtained by successively applying Lemma 3.1 and Ito's lemma to (3.1).

Proposition 3.2. *Under Assumptions 3.1-3.5, we have*

$$
\begin{aligned}
\hat{\beta}_v - 1 &\sim_p \Delta \frac{\int_0^T (V_t - \overline{V}_T) dV_t}{\int_0^T (V_t - \overline{V}_T)^2 dt}, \\
\hat{\beta}_x - 1 &\sim_p \Delta \frac{\int_0^T (V_t - \overline{V}_T) dV_t + (1/6)[V]_T}{\int_0^T (V_t - \overline{V}_T)^2 dt}, \\
\hat{\beta}_y - 1 &\sim_p \Delta \frac{\int_0^T (V_t - \overline{V}_T) dV_t + (1/6)[V]_T - (2\delta/\Delta^2)\int_0^T V_t^2 dt}{\int_0^T (V_t - \overline{V}_T)^2 dt}
\end{aligned}
$$

and

$$
\hat{\beta}_z^{(k)} - 1 \sim_p (\hat{\beta}_z - 1) + k(\hat{\beta}_v - 1).
$$

Remark 3.2. (a) As explained in Remark 3.1 (a), (x_i) has more smoother sample paths than (v_i), and hence, we have (3.2). In Proposition 3.2, we have $\hat{\beta}_v < \hat{\beta}_x$ which implies that (x_i) tends to have more persistent sample paths than (v_i). This result is a consequence of (3.2). Moreover, $\hat{\beta}_y$ is downward biased with $\hat{\beta}_y < \hat{\beta}_x$ which is induced by the errors in (y_i).

(b) Note that Assumptions 3.3-3.5 do not necessarily imply $\delta/\Delta^2 \to 0$. Therefore, the speeds of $\delta \to 0$ and $\Delta \to 0$ are important in the asymptotic negligibility of the estimation errors of (y_i). In particular, if $\delta/\Delta^2 \to 0$ sufficiently quickly, then the errors of (y_i) become asymptotically negligible, and hence, we may have $\hat{\beta}_y - 1 \sim_p \hat{\beta}_x - 1$. In Section 5.1, we analyze the asymptotics negligibility of the estimation errors of (y_i) for more general class of estimators, including the OLS and IV estimators, under the presence of leverage effects.

3.2. Long Span Asymptotics

The primary asymptotics in Proposition 3.2 do not require $T \to \infty$. In particular, if T is fixed, then $(N/T)(\hat{\beta}_z - 1) = (1/\Delta)(\hat{\beta}_z - 1)$ is random for all $z = v, x, y$, and is determined by a particular realization of the underlying variance process V. Under the fixed T asymptotic scheme, the law of motion of V is less important. In particular, the results in Proposition 3.2 require neither certain moment conditions nor stationarity. However, the underlying probabilistic structure of V is crucial in the development of the large T asymptotics.

In our long span asymptotics, we only consider a stationary diffusion V to effectively analyze consequences of fat tails in the volatility regressions. Under Assumption 2.1, the diffusion V is recurrent if and only if $s(\underline{v}) = -\infty$ and $s(\overline{v}) = \infty$, where s is the scale function defined in (2.5). A diffusion which is not recurrent is said to be transient. Furthermore, the recurrent diffusion V becomes positive recurrent or null recurrent, depending upon whether the speed density m in (2.6) is integrable on $\mathcal{D}$ or not. Positive recurrent diffusions have time invariant distributions, and if they are started from the time invariant distributions they become stationary. The time invariant density of the positive recurrent diffusion V is given by $\pi(v) = m(v)/\int_{\mathcal{D}} m(v)dv$. Therefore, conditions on unconditional moments are characterized by corresponding m-integrability conditions. For instance, $\mathbb{E}(f(V_t)) < \infty$ if and only if f is m-integrable, since $\mathbb{E}(f(V_t)) = \int_{\mathcal{D}} f(v)\pi(v)dv$ and $\pi(v) = m(v)/\int_{\mathcal{D}} m(v)dv$ with $\int_{\mathcal{D}} m(v)dv < \infty$.

Since we allow fat tails, we consider not only integrable functions but also nonintegrable functions with respect to the speed density m of V. We will not require any regularity conditions for m-integrable functions. To effectively analyze m-nonintegrable functions, however, we need some regularity conditions. Following Kim and Park (2017), it will be maintained throughout the paper that all m-nonintegrable functions f are m-regularly varying, i.e., mf is regularly varying on $\mathcal{D}$. For a m-nonintegrable function f, we say that f is m-strongly nonintegrable if $f\ell$ is not m-integrable for any slowly varying function ℓ on $\mathcal{D}$. On the other hand, we say that f is m-nearly integrable if $f\ell$ is m-integrable for some slowly varying function ℓ on $\mathcal{D}$.

We assume that

Assumption 3.6. *(a) s' is regularly varying or rapidly varying with index $c \neq -1$, (b) σ^2 is regularly varying, and (c) $f = \sigma^2, \iota^2$ is either m-integrable or m-strongly* *nonintegrable.*

Assumption 3.6 (a) and (b) appear in Kim and Park (2018), and are mild enough to include most diffusion processes used in practice. The reader is also referred to Bingham et al. (1993) for more discussions about the regularly and rapidly varying functions. In Assumption 3.6 (c), we assume that σ^2 and ι^2 are m-strongly nonintegrable as long as they are not m-integrable. This assumption is a technical condition to simplify our discussions below. Our subsequent theory can also be developed under the m-near integrablility at the cost of more involved analysis (see Kim and Park (2017, 2018) for the related discussions).

In the following, we let $f_s = f \circ s^{-1}$ for any function f on $\mathcal{D}$ other than m.[4] Moreover, for a regularly varying function f on $\mathbb{R}$, we define its limit homogeneous function $\overline{f}$ as $f(\lambda v)/f(\lambda) \to \overline{f}(v)$ as $\lambda \to \infty$ for all $v \neq 0$.

We define numerical sequences p_T and q_T as

$$p_T = \begin{cases} T & \text{if } \sigma^2 \text{ is } m\text{-integrable} \\ T^2(m_s\sigma_s^2)(T) & \text{if } \sigma^2 \text{ is } m\text{-strongly nonintegrable} \end{cases}$$
$$q_T = \begin{cases} T & \text{if } \iota^2 \text{ is } m\text{-integrable} \\ T^2(m_s\iota_s^2)(T) & \text{if } \iota^2 \text{ is } m\text{-strongly nonintegrable,} \end{cases}$$

and let

$$P = \begin{cases} \mathbb{E}(\sigma^2(V_t)) & \text{if } \sigma^2 \text{ is } m\text{-integrable} \\ \int_0^\tau \overline{m_s\sigma_s^2}(B_t)dt & \text{if } \sigma^2 \text{ is } m\text{-strongly nonintegrable} \end{cases}$$
$$Q = \begin{cases} \mathbb{E}(V_t^2) & \text{if } \iota^2 \text{ is } m\text{-integrable} \\ \int_0^\tau \overline{m_s\iota_s^2}(B_t)dt & \text{if } \iota^2 \text{ is } m\text{-strongly nonintegrable} \end{cases}$$
$$S = \begin{cases} \mathbb{E}(V_t^2) - (\mathbb{E}(V_t))^2 & \text{if } \iota^2 \text{ is } m\text{-integrable} \\ \int_0^\tau \overline{m_s\iota_s^2}(B_t)dt & \text{if } \iota^2 \text{ is } m\text{-strongly nonintegrable,} \end{cases}$$

where B is Brownian motion and $\tau = \inf\{t \,|\, L(t,0) > 1/\int_{\mathcal{D}} m(v)dv\}$ with Brownian local time $L(\cdot,0)$ of B at the origin (i.e., $L(t,0) = \lim_{\epsilon\to 0}(2\epsilon)^{-1}\int_0^t 1\{|B_s| < \epsilon\}ds$). Under Assumption 3.6, both (p_T, q_T) and (P, Q, S) are well defined (see Kim and

[4] In Section 2.1, m_s is defined as $m_s = (m/s') \circ s^{-1}$ which is the speed density of natural scale diffusion $V^s = s(V)$ of the underlying diffusion V.

Park (2017)).

Lemma 3.3. *Let Assumption 3.6 hold. Then $Tp_T/q_T \to \infty$ and*

$$\frac{1}{p_T}[V]_T \to_d P, \qquad \frac{1}{p_T}\int_0^T (V_t - \overline{V}_T) dV_t \to_d -\frac{P}{2},$$
$$\frac{1}{q_T}\int_0^T V_t^2 dt \to_d Q, \qquad \frac{1}{q_T}\int_0^T (V_t - \overline{V}_T)^2 dt \to_d S$$

as $T \to \infty$.

Under the m-integrability of $f = \iota^2, \sigma^2$, Lemma 3.3 becomes a standard law of large numbers of stationary diffusions. However, if $f = \iota^2, \sigma^2$ is not m-integrable, the standard limit theory is not applicable and we have completely different limit theory. In particular, the limit distribution $\int_0^\tau \overline{m_s f_s}(B_t)dt$ is not Gaussian and is highly nonstandard. Moreover, the normalizing sequence $T^2(m_s f_s)(T)$ diverges faster than T since the function $m_s f_s$ becomes a regularly varying function with index $c > -1$ as long as f is not m-integrable and Assumption 3.6 holds. The reader is referred to Kim and Park (2017) for more detailed discussions about the asymptotics of diffusion functionals.

The long span asymptotics for $\hat{\beta}_z$ follow immediately from Proposition 3.2 with Lemma 3.3.

Theorem 3.4. *Let Assumptions 3.1-3.6 hold. As $\delta/\Delta, \Delta \to 0$ and $T \to \infty$, we have*

$$\hat{\beta}_v - 1 \sim_d -\Delta \frac{p_T}{q_T}\frac{P}{2S}, \qquad \hat{\beta}_x - 1 \sim_d -\Delta \frac{p_T}{q_T}\frac{P}{3S}, \qquad \hat{\beta}_y - 1 \sim_d -\Delta \frac{p_T}{q_T}\frac{P}{3S} - \frac{\delta}{\Delta}\frac{2Q}{S}.$$

As shown in Lemma 3.3 that P, Q and S become constants only when both ι^2 and σ^2 are m-integrable. The relation $\sim_d$ in Theorem 3.4 becomes $\sim_p$ if P, Q and S are all constants. On the other hand, if ι^2 and σ^2 are not m-integrable, then P, Q and S remain random. In this case, Theorem 3.4 implies that $\hat{\beta}_z - 1$ is random for all small Δ.

Remark 3.3. The results in Theorem 3.4 can be applied to a broad class of volatility processes used in the literature.

(a) If both σ^2 and ι^2 are m-integrable, then $p_T = q_T = T$ and

$$\hat{\beta}_v - 1 \sim_p -\Delta \frac{\mathbb{E}(\sigma^2(V_t))}{2Var(V_t)}, \quad \hat{\beta}_x - 1 \sim_p \frac{2}{3}(\hat{\beta}_v - 1), \quad \hat{\beta}_y - 1 \sim_p (\hat{\beta}_x - 1) - \frac{\delta}{\Delta}\frac{2\mathbb{E}(V_t^2)}{Var(V_t)}.$$

(b) For a stationary Ornstein-Uhlenbeck process V, given as

$$dV_t = \kappa(\mu - V_t)dt + \sigma dW_t,$$

we have $\mathbb{E}(\sigma^2(V_t)) = \sigma^2$, $Var(V_t) = \sigma^2/(2\kappa)$ and $\mathbb{E}(V_t^2) = \sigma^2/(2\kappa) + \mu^2$. Therefore,

$$\hat{\beta}_v - 1 \sim_p -\Delta\kappa, \quad \hat{\beta}_x - 1 \sim_p -\Delta\frac{2}{3}\kappa, \quad \hat{\beta}_y - 1 \sim_p -\Delta\frac{2}{3}\kappa - 2\frac{\delta}{\Delta}\left(1 + \frac{2\kappa\mu^2}{\sigma^2}\right). \tag{3.6}$$

(c) Let V be a stationary GARCH diffusion (2.7) with $\sigma^2 < 2\kappa$ so that $\mathbb{E}(V_t^2) < \infty$. In this case, we have $\mathbb{E}(\sigma^2(V_t)) = \sigma^2\mathbb{E}(V_t^2)$, $Var(V_t) = \mu^2\sigma^2/(2\kappa - \sigma^2)$ and $\mathbb{E}(V_t^2) = 2\kappa\mu^2/(2\kappa - \sigma^2)$, and hence, Theorem 3.4 implies

$$\hat{\beta}_v - 1 \sim_p -\Delta\kappa, \qquad \hat{\beta}_x - 1 \sim_p -\Delta\frac{2}{3}\kappa, \qquad \hat{\beta}_y - 1 \sim_p -\Delta\frac{2}{3}\kappa - 4\frac{\delta}{\Delta}\frac{\kappa}{\sigma^2}. \tag{3.7}$$

It is interesting to note that the results (3.7) are the same as (2.8) for population regressions derived by Andersen et al. (2004).

(d) Let V be a stationary GARCH diffusion (2.7) with $2\kappa < \sigma^2$ so that $\mathbb{E}(V_t^2) = \mathbb{E}(\sigma^2(V_t)) = \infty$. In this case, $p_T = \sigma^2 q_T$ and $P = Q = S$, and therefore, we have

$$\hat{\beta}_v - 1 \sim_p -\Delta\frac{1}{2}\sigma^2, \qquad \hat{\beta}_x - 1 \sim_p -\Delta\frac{1}{3}\sigma^2, \qquad \hat{\beta}_y - 1 \sim_p -\Delta\frac{1}{3}\sigma^2 - 2\frac{\delta}{\Delta}. \tag{3.8}$$

Under $\mathbb{E}(V_t^2) < \infty$, as shown in Remark 3.3 (c), the limits of $(\hat{\beta}_z - 1)/\Delta$ are mainly determined by the mean reversion parameter κ in the drift function $\mu(v)$. Under $\mathbb{E}(V_t^2) = \infty$, the limits $(\hat{\beta}_z - 1)/\Delta$ are still constant, but they are determined by the diffusion parameter σ^2 in the diffusion function $\sigma^2(v)$.

We also note that GARCH diffusion is a special example that $(\hat{\beta}_z - 1)/\Delta$ has a degenerated constant limit even under $\mathbb{E}(V_t^2) = \infty$, which is induced by the relationship $v^2 \propto \sigma^2(v)$ between the quadratic function v^2 and the diffusion function $\sigma^2(v)$. For any other models which do not satisfy $\iota^2(v) \propto \sigma^2(v)$ asymptotically, $(\hat{\beta}_z - 1)$ has a random limit, after proper normalization, as long as $\mathbb{E}(V_t^2) = \infty$ or $\mathbb{E}(\sigma^2(V_t)) = \infty$. This is the case for the CEV process considered below.

(e) Let V be a stationary CEV process

$$dV_t = \kappa(\mu - V_t)dt + \sigma V_t^\gamma dW_t.$$

If $\kappa, \mu, \sigma > 0$ and $1 < \gamma < 3/2$, then $\mathbb{E}(V_t^2) = \infty$ and $\mathbb{E}(\sigma^2(V_t)) = \infty$ since $m(v) \sim v^{-2\gamma}$ as $v \to \infty$. For the CEV process, we have

$$p_T = \sigma T^2(m_s \iota_s^{2\gamma})(T), \qquad q_T = T^2(m_s \iota_s^2)(T),$$
$$P = \int_0^\tau \overline{m_s \iota_s^{2\gamma}}(B_t)dt, \qquad Q = S = \int_0^\tau \overline{m_s \iota_s^2}(B_t)dt,$$

where $p_T/q_T = \sigma \iota_s^{2\gamma-2}(T) \to \infty$ as $T \to \infty$, since $\gamma > 1$ and $\iota_s = s^{-1}$ is monotonically increasing by the recurrence property. Clearly, $P \neq S$ for any $\gamma \in (1, 3/2)$, and hence, P/S remains random unlike the GARCH diffusion. Therefore, $\hat{\beta}_z - 1$ has random limit for all sufficiently small Δ.

(f) Our example in Remark 3.3 (d) should be contrasted to the limit theory for the sample autocorrelations of GARCH(1,1) processes with fat tails obtained in Mikosch and Starica (2000). Let

$$X_i = \sigma_i Z_i \quad \text{with} \quad \sigma_i^2 = \alpha_0 + \beta_1 \sigma_{i-1}^2 + \alpha_1 X_{i-1}^2 \quad \text{for} \quad i = 1, 2, \cdots, N,$$

where (Z_i) is a sequence of i.i.d. symmetric random variables with $\mathbb{E}Z_i^2 = 1$. Under some assumptions, which imply that the vector (X_i, σ_i) is regularly varying with index $p > 0$, it is shown that for $p \in (0, 4)$ the variance process (σ_i^2) has unbounded variance and satisfies, for any $k \geq 1$,

$$\left(\frac{\sum_{i=1}^{N-k} X_i^2 X_{i+k}^2}{\sum_{i=1}^N X_i^4} - 1, \frac{\sum_{i=1}^{N-k} \sigma_i^2 \sigma_{i+k}^2}{\sum_{i=1}^N \sigma_i^4} - 1 \right) \sim_d \left(\frac{\Sigma_{1,X^2} - \Sigma_{0,X^2}}{\Sigma_{0,X^2}}, \frac{\Sigma_{1,\sigma^2} - \Sigma_{0,\sigma^2}}{\Sigma_{0,\sigma^2}} \right),$$

where the limit distribution is nondegenerated since the vector $(\Sigma_{m,X^2}, \Sigma_{m,\sigma^2})_{m=0,1}$ is $p/2$-stable. This contrasts with our result for a GARCH diffusion with unbounded variance (see Remark 3.3 (d)), in which $(\hat{\beta}_z - 1)/\Delta$ has a constant limit for $z = v, x$. We think that the difference between our results and those of Mikosch and Starica

(2000) is due to the fact that we allow $\Delta \to 0$. We conjecture that the result would be the same when Δ is fixed.

4. Instrumental Variable Estimations

In this section we study various IV estimators of $\beta_z^{(k)}$ in (2.3). When one has a model like

$$y_i = \alpha^0 + \beta^0 x_i + u_i$$

where x_i has fat tails while u_i is i.i.d. with possible fat tails, it is well known that in general the OLS estimator of β^0 is consistent. However, the OLS is not necessarily efficient and its asymptotic distribution could be non-Gaussian. An alternative method that could lead to more efficient estimators or asymptotically Gaussian ones is to consider a signed power estimator defined as[5]

$$\tilde{\beta} = \frac{\sum \operatorname{sign}(x_i) \mid x_i \mid^c (y_i - \bar{y})}{\sum \operatorname{sign}(x_i) \mid x_i \mid^c (x_i - \bar{x})}. \tag{4.1}$$

The reader is referred to Samorodnitsky et al. (2007) for the asymptotics of OLS and the signed power estimator.

One can easily prove that this estimator is indeed the empirical counterpart of the IV estimator defined by

$$\mathbb{E}\left[\begin{pmatrix} 1 \\ \operatorname{sign}(x_i)|x_i|^c \end{pmatrix} (y_i - \alpha^0 - \beta^0 x_i)\right] = 0.$$

Typically, the constant c is smaller than one in order to reduce the tails of moments involved in the estimation method. An extreme case is the Cauchy estimator which corresponds to $c = 0$, that is the instrument equals the sign of x_i.

Consequently, we study in the following subsection IV estimators of $\beta_z^{(k)}$ in (2.3) which have the form

$$\tilde{\beta}_z^{(k)} = \frac{\sum_{i=k+1}^{N-1} r(z_{i-k})(z_{i+1} - \overline{z}_N)}{\sum_{i=k+1}^{N-1} r(z_{i-k})(z_{i-k} - \overline{z}_N)},$$

[5] If the variable x does not change a sign like a volatility measure, the instrument should be $\operatorname{sign}(x_i - \bar{x})|x_i - \bar{x}|^c$.

where we use a functional transformation $r(z_{i-k})$ of z_{i-k} as an instrument. We prove below that the IV estimator, with a proper choice of instrument, is robust to fat tails.

Interestingly, Jean-Marie Dufour used in several studies sign-based methods for inference purposes, especially for exact inference in finite sample. In particular, he used such approach in Coudin and Dufour (2009, 2017) in order to provide inference about the slope parameter in a linear regression model without making moment restrictions on the disturbance errors and therefore allowing for fat tails. The assumption made in these papers is a median restriction on the errors conditional on the explanatory variables. In other words, we are using the same approach with a slightly different framework because we assume that the (conditional) first moment of the errors exists and equals zero but we do not make assumptions on higher moments.

4.1. IV Estimator $\tilde{\beta}_z^{(k)}$ with a Current Instrument

Let r be continuously differentiable, and define $r_1(z) = \int_{z_0}^{z} r(x)dx$ for some $z_0 \in \mathcal{D}$. Then, by Taylor expansion, we have

$$\begin{aligned}\sum_{j=0}^{k}(r_1(z_{N-j}) - r_1(z_{1+j})) &= \sum_{i=k+1}^{N-1}(r_1(z_{i+1}) - r_1(z_{i-k})) \\ &= \sum_{i=k+1}^{N-1} r(z_{i-k})(z_{i+1} - z_{i-k}) + \frac{1}{2}\sum_{i=k+1}^{N-1} r'(z_{i-k}^*)(z_{i+1} - z_{i-k})^2\end{aligned}$$

for $(z_i^*)_{i=1}^{N-1}$ such that $z_i^* \in [z_{i-k}, z_{i+1}]$. Using the expansion, we may rewrite $\tilde{\beta}_z^{(k)}$ as

$$\tilde{\beta}_z^{(k)} - 1 = \frac{\sum_{j=0}^{k}(r_1(z_{N-j}) - r_1(z_{1+j}))}{\sum_{i=k+1}^{N-1} r(z_{i-k})(z_{i-k} - \overline{z}_N)} - \frac{1}{2}\frac{\sum_{i=k+1}^{N-1} r'(z_{i-k}^*)(z_{i+1} - z_{i-k})^2}{\sum_{i=k+1}^{N-1} r(z_{i-k})(z_{i-k} - \overline{z}_N)}. \tag{4.2}$$

As in Lemma 3.1, we may obtain the continuous time approximation for each term in (4.2). For the approximation, we require

Assumption 4.1. *(a) r is three times continuously differentiable on $\mathbb{R}$ with $r'(z) > 0$ for all $z \in \mathbb{R}$, and (b) r and its derivatives are all majorized by the function ω in Assumption 3.1.*

The role of Assumption 4.1 is similar to Assumption 3.1, and make it convenient to develop the continuous time approximations if combined with the conditions on δ, Δ and T in Assumptions 3.2-3.5.

Proposition 4.1. *Let Assumptions 3.1-3.5 and 4.1 hold.*

(a) For $k \geq 0$, we have

$$\sum_{j=0}^{k}(r_1(z_{N-j}) - r_1(z_{1+j})) \sim_p (1+k)(r_1(V_T) - r_1(V_0)),$$
$$\sum_{i=k+1}^{N-1} r(z_{i-k})(z_{i-k} - \overline{z}_N)\Delta \sim_p \int_0^T r(V_t)(V_t - \overline{V}_T)dt$$

for $z = v, x, y$.

(b) For $k \geq 0$, we have

$$\sum_{i=k+1}^{N-1} r'(z_{i-k}^*)(z_{i+1} - z_{i-k})^2 \sim_p \sum_{i=1}^{N-1} r'(z_i)(z_{i+1} - z_i)^2 + k\sum_{i=1}^{N-1} r'(v_i)(v_{i+1} - v_i)^2$$

for $z = v, x, y$, and

$$\sum_{i=1}^{N-1} r'(z_i)(z_{i+1} - z_i)^2 \sim_p \begin{cases} \int_0^T r'(V_t)d[V]_t, & \text{for } z = v \\ (2/3)\int_0^T r'(V_t)d[V]_t, & \text{for } z = x \\ (2/3)\int_0^T r'(V_t)d[V]_t + (4\delta/\Delta^2)\int_0^T r'(V_t)V_t^2 dt, & \text{for } z = y \end{cases}$$

(c) We have

$$\tilde{\beta}_v - 1 \sim_p \Delta \frac{\int_0^T r(V_t)dV_t}{\int_0^T r(V_t)(V_t - \overline{V}_T)dt},$$
$$\tilde{\beta}_x - 1 \sim_p \Delta \frac{\int_0^T r(V_t)dV_t + (1/6)\int_0^T r'(V_t)d[V]_t}{\int_0^T r(V_t)(V_t - \overline{V}_T)dt},$$
$$\tilde{\beta}_y - 1 \sim_p \Delta \frac{\int_0^T r(V_t)dV_t + (1/6)\int_0^T r'(V_t)d[V]_t - (2\delta/\Delta^2)\int_0^T r'(V_t)V_t^2 dt}{\int_0^T r(V_t)(V_t - \overline{V}_T)dt}$$

and

$$\tilde{\beta}_z^{(k)} - 1 \sim_p (\tilde{\beta}_z - 1) + k(\tilde{\beta}_v - 1).$$

We note that if $r(z) = z$, then $\tilde{\beta}_z^{(k)}$ becomes the OLS estimator $\hat{\beta}_z^{(k)}$ in Section

3. Proposition 4.1 (a) and (b) are generalizations of Lemma 3.1 (a) and (b), respectively. Similarly, Proposition 4.1 (c) is a generalization of Proposition 3.2. Moreover, Remarks 3.1 and 3.2 remain valid. We also note that if $r(z_{i-k} - \overline{z}_N)$ is used as an instrument instead of $r(z_{i-k})$, then the results in Proposition 4.1 (c) holds with $r(V_t - \overline{V}_T)$ and $r'(V_t - \overline{V}_T)$ in place of $r(V_t)$ and $r'(V_t)$, respectively.

Now we develop the large T asymptotics of $\tilde{\beta}_z$. To effectively control the fat tails in V, we impose the following conditions on r.

Assumption 4.2. *The function $r : \mathcal{D} \to \mathbb{R}$ satisfies that $\mathbb{E}[r(V_t)], \mathbb{E}[r(V_t)V_t], \mathbb{E}[r'(V_t)V_t^2]$, and $\mathbb{E}[r'(V_t)\sigma^2(V_t)]$ are all bounded, and $\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t] \neq 0$.*

Assumption 4.2 provides simple sufficient conditions to ensure that the IV estimator $\tilde{\beta}_z$ has a constant limit involving parameters in μ and σ^2. An example of r satisfying Assumption 4.2 is $r(v) = \arctan(v)$. Clearly, r is monotonically increasing, bounded and continuously differentiable with $r'(v) = 1/(1 + v^2)$. Therefore, Assumption 4.2 holds if $\mathbb{E}|V_t| < \infty$ and $\mathbb{E}[r'(V_t)\sigma^2(V_t)] < \infty$. When $\mathcal{D} = (0, \infty)$ and $r(v) = \arctan(v)$, we have $\mathbb{E}[r'(V_t)\sigma^2(V_t)] < \infty$ as long as $\sigma^2(v)/v^3 = O(v^\epsilon)$ as $v \to \infty$ for some $\epsilon > 0$.

Theorem 4.2. *Let Assumptions 3.1-3.5 and 4.1-4.2 hold. If $\mathbb{E}|V_t| < \infty$, then*

$$\begin{aligned}
\tilde{\beta}_v - 1 &\sim_p -\Delta \frac{1}{2} \frac{\mathbb{E}[r'(V_t)\sigma^2(V_t)]}{\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t]}, \\
\tilde{\beta}_x - 1 &\sim_p -\Delta \frac{1}{3} \frac{\mathbb{E}[r'(V_t)\sigma^2(V_t)]}{\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t]}, \\
\tilde{\beta}_y - 1 &\sim_p -\Delta \frac{1}{3} \frac{\mathbb{E}[r'(V_t)\sigma^2(V_t)]}{\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t]} - \frac{\delta}{\Delta} \frac{2\mathbb{E}[r'(V_t)V_t^2]}{\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t]}.
\end{aligned}$$

As well expected, $\tilde{\beta}_z - 1$ has a well defined constant limit under the moment conditions in Assumption 4.2. For a given parametric diffusion model, we may explicitly compute the limit of $\tilde{\beta}_z - 1$. As an example, we consider a stationary diffusion V defined on $\mathcal{D} = (0, \infty)$ having a linear drift

$$dV_t = \kappa(\mu - V_t)dt + \sigma(V_t)dW_t \tag{4.3}$$

with $\mathbb{E}(V_t) = \mu$ and $\sigma^2(v)/v^2 = O(1)$ as $v \to \infty$.

Corollary 4.3. *Let r be a bounded function satisfying Assumption 4.2 for a given V in (4.3). Then we have*

$$\frac{\mathbb{E}[r'(V_t)\sigma^2(V_t)]}{\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t]} = 2\kappa.$$

Moreover, if V in (4.3) is a GARCH diffusion with $\sigma^2(v) = \sigma^2 v^2$, then

$$\frac{\mathbb{E}[r'(V_t)V_t^2]}{\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t]} = 2\frac{\kappa}{\sigma^2}.$$

For a GARCH diffusion, it follows immediately from Theorem 4.2 and Corollary 4.3 that

$$\tilde{\beta}_v - 1 \sim_p -\Delta\kappa, \qquad \tilde{\beta}_x - 1 \sim_p -\Delta\frac{2}{3}\kappa, \qquad \tilde{\beta}_y - 1 \sim_p -\Delta\frac{2}{3}\kappa - 4\frac{\delta}{\Delta}\frac{\kappa}{\sigma^2} \tag{4.4}$$

hold regardless of the finiteness of $\mathbb{E}(V_t^2)$. In contrast, the OLS estimates $\hat{\beta}_z$ have different limits depending upon $\mathbb{E}(V_t^2) < \infty$ holds or not (see the discussions in Remark 3.3 (c) and (d)). Moreover, the limits of $\tilde{\beta}_z$ in (4.4) are equivalent to those of the OLS estimates $\hat{\beta}_z$ in (3.6), which are obtained under $\mathbb{E}(V_t^2) < \infty$. Therefore, we may say that the instrumental variable approach can effectively control the fat tails as long as r is appropriately chosen.

If the transformation r satisfies some additional integrability conditions, we may obtain the asymptotic normality of the IV estimator. For the asymptotic normality, we use the asymptotics of zero functionals (see, e.g., Mandl (1968); van der Vaart and van Zanten (2005)) so that we have

$$\sqrt{T}\left(\frac{1}{T}\int_0^T (r'\sigma^2)(V_t)dt - \mathbb{E}[r'(V_t)\sigma^2(V_t)]\right) \to_d \mathbb{N}(0, \Sigma_r), \tag{4.5}$$

provided that the asymptotic variance

$$\Sigma_r = 4\left(\int_{\mathcal{D}} m(v)dv\right)\left[\int_{\mathcal{D}}\left(\int_{\underline{v}}^{v}\left\{(r'\sigma^2)(v) - \mathbb{E}[r'(V_t)\sigma^2(V_t)]\right\}\pi(u)du\right)^2 ds(v)\right]$$

is finite, where m, π and s are the speed density, time invariant distribution and scale function, respectively. Therefore, if r is appropriately chosen such that $\Sigma_r < \infty$, we may deduce from Proposition 4.1 (c), Assumption 4.2 and (4.5) that

$$\sqrt{T}\left(\tilde{\beta}_v - 1 + \frac{\Delta}{2}\frac{\mathbb{E}[r'(V_t)\sigma^2(V_t)]}{\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t]}\right) \to_d (2\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t])^{-1}\,\mathbb{N}(0,\Sigma_r),$$
$$\sqrt{T}\left(\tilde{\beta}_x - 1 + \frac{\Delta}{3}\frac{\mathbb{E}[r'(V_t)\sigma^2(V_t)]}{\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t]}\right) \to_d (3\mathbb{E}[r(V_t)V_t] - \mathbb{E}[r(V_t)]\mathbb{E}[V_t])^{-1}\,\mathbb{N}(0,\Sigma_r),$$

and $\tilde{\beta}_y - 1$ has the same asymptotic distribution as $\tilde{\beta}_x - 1$ if $\delta/\Delta^2 \to 0$. On the other hand, if r does not satisfy $\Sigma_r < \infty$, then (4.5) does not hold, and the limit distributions of $\tilde{\beta}_z - 1$ are not Gaussian (see Theorem 3.6 of Kim and Park (2017)).

Heuristically, we may consider the Cauchy estimator by using $r(z - \overline{z}_N)$, where $r(z) = \mathrm{sign}(z)$, as an instrument in $\tilde{\beta}_z$. Clearly, r is not differentiable, and hence, our results Proposition 4.1 and Theorem 4.2 are not directly applicable. By the standard approximation method with Tanaka's formula, however, we may obtain the asymptotics of the Cauchy estimator. Given Proposition 4.1 (c), we conjecture that

$$\begin{aligned}
\tilde{\beta}_v - 1 &\sim_p \Delta\frac{\int_0^T \mathrm{sign}(V_t - \overline{V}_T)dV_t}{\int_0^T |V_t - \overline{V}_T|dt},\\
\tilde{\beta}_x - 1 &\sim_p \left(\tilde{\beta}_v - 1\right) + \frac{\Delta}{3}\frac{\sigma^2(\overline{V}_T)L_V(T,\overline{V}_T)}{\int_0^T |V_t - \overline{V}_T|dt},\\
\tilde{\beta}_y - 1 &\sim_p \left(\tilde{\beta}_x - 1\right) - \frac{4\delta}{\Delta}\frac{(\overline{V}_T)^2 L_V(T,\overline{V}_T)}{\int_0^T |V_t - \overline{V}_T|dt},
\end{aligned}$$

where $L_V(\cdot, v)$ is the local time of V at $v \in \mathcal{D}$, defined as $L_V(T,v) = \lim_{\epsilon\to 0}(2\epsilon)^{-1}\int_0^T 1\{|V_t - v| < \epsilon\}dt$. The large T asymptotics then follow immediately from the law of large numbers, and they are given by

$$\tilde{\beta}_v - 1 \sim_p -\Delta\frac{\sigma^2(\mu)\pi(\mu)}{\mathbb{E}|V_t - \mu|}, \quad \tilde{\beta}_x - 1 \sim_p -\Delta\frac{2\sigma^2(\mu)\pi(\mu)}{3\mathbb{E}|V_t - \mu|}, \quad \tilde{\beta}_y - 1 \sim_p (\tilde{\beta}_x - 1) - \frac{4\delta}{\Delta}\frac{\mu^2\pi(\mu)}{\mathbb{E}|V_t - \mu|},$$

since $\overline{V}_T \to_p \mathbb{E}[V_t] = \mu$, $T^{-1}L_V(T,\overline{V}_T) \to_p \pi(\mu)$, $T^{-1}\int_0^T |V_t - \overline{V}_T|dt \to_p \mathbb{E}|V_t - \mu|$ and

$$\frac{1}{T}\int_0^T \mathrm{sign}(V_t - \overline{V}_T)dV_t \to_p -\sigma^2(\mu)\pi(\mu).$$

If V is a GARCH diffusion, it can be shown as in Corollary 4.3 that

$$\frac{\sigma^2(\mu)\pi(\mu)}{\mathbb{E}|V_t-\mu|}=\kappa, \qquad \frac{\mu^2\pi(\mu)}{\mathbb{E}|V_t-\mu|}=\frac{\kappa}{\sigma^2}.$$

Therefore, we conjecture that (4.4) holds even when the Cauchy estimator is used. We will formally analyze the asymptotics, under a fixed Δ, of the Cauchy estimator in Section 5.2.

4.2. IV Estimator $\breve{\beta}_z^{(k)}$ with a Lagged Instrument

In the GARCH diffusion case, (4.4) means that the proposed IV estimator converges to the object of interest when one uses the spot variance while one gets a bias estimation when one uses integrated or realized variance. The reason is that the integrated and realized variances are ARMA(1,1) processes and therefore this IV estimator converges to the first order autocorrelation (when the second moment of these variables are bounded). As mentionned above, a solution to this problem is to consider the multiperiod moment restriction (2.11), which in turn corresponds to consider a lagged instrument in the estimation of $\beta_z^{(k)}$ in (2.3). More precisely, in this subsection we study the estimator $\breve{\beta}_z^{(k)}$ defined as

$$\breve{\beta}_z^{(k)}=\frac{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i+1}-\overline{z}_N)}{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k}-\overline{z}_N)}.$$

In other words, the IV estimator $\tilde{\beta}_z^{(k)}$ studied in the previous subsection uses $r(z_{i-k})$ as an instrument for $(z_{i-k}-\overline{z}_N)$, whereas $\breve{\beta}_z^{(k)}$ employes $r(z_{i-k-1})$ as an instrument for the same object $(z_{i-k}-\overline{z}_N)$.

For the asymptotics, we write

$$\begin{aligned}\breve{\beta}_z^{(k)}-1&=\frac{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i+1}-z_{i-k})}{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k}-\overline{z}_N)}\\ &=\frac{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i+1}-z_{i-k-1})}{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k}-\overline{z}_N)}-\frac{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k}-z_{i-k-1})}{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k}-\overline{z}_N)}\\ &\equiv \phi_z^{(k)}-\psi_z^{(k)}.\end{aligned} \tag{4.6}$$

For the denominator of $\phi_z^{(k)}$ and $\psi_z^{(k)}$ with a fixed $k \geq 0$, we may show that

$$\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k} - \overline{z}_N)\Delta \sim_p \sum_{i=k+2}^{N-1} r(z_{i-k})(z_{i-k} - \overline{z}_N)\Delta \sim_p \int_0^T r(V_t)(V_t - \overline{V}_T)dt$$

as long as δ/Δ and Δ are sufficiently small. We then may deduce from Proposition 4.1 with (4.2) that

$$\phi_z^{(k)} \sim_p \Delta \frac{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i+1} - z_{i-k-1})}{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k-1} - \overline{z}_N)\Delta} \sim_p \tilde{\beta}_z^{(1+k)} - 1$$

and

$$\psi_z^{(k)} \sim_p \Delta \frac{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k} - z_{i-k-1})}{\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k-1} - \overline{z}_N)\Delta} \sim_p \tilde{\beta}_z^{(0)} - 1$$

as long as δ/Δ and Δ are sufficiently small. We formally have

Theorem 4.4. *Let Assumptions 3.1-3.5 and 4.1 hold. For $z = v, x, y$, we have*

$$\check{\beta}_z^{(k)} - 1 \sim_p (\tilde{\beta}_z^{(1+k)} - 1) - (\tilde{\beta}_z^{(0)} - 1) \sim_p (1+k)(\tilde{\beta}_v - 1).$$

Unlike $\hat{\beta}_z$ and $\tilde{\beta}_z$, the limits of $\check{\beta}_z - 1$ are given by $\tilde{\beta}_v - 1$ regardless of $z = v, x, y$. Consequently, if V is a linear drift diffusion in (4.3), then $(\check{\beta}_z - 1)/\Delta \to_p -\kappa$ for all $z = v, x, y$, rather than (4.4). Therefore, we may say that the IV estimator $\check{\beta}_z$ is a consistent estimator for the mean reversion parameter κ of linear drift diffusions, and is robust to not only fat tails in V but also errors $(v_i - x_i)$ and $(v_i - y_i)$ in, respectively, the integrated variance and realized variance.

For the linear transformation $r(z) = z$, we may easily see that $\tilde{\beta}_z = \hat{\beta}_z$. In this case, the IV estimator $\check{\beta}_z$ becomes a simple IV estimator with an instrument z_{i-1} for $(z_i - \overline{z}_N)$, and it follows from Theorem 4.4 that

$$\check{\beta}_z^{(k)} - 1 \sim_p (1+k)(\hat{\beta}_v - 1) \tag{4.7}$$

for $z = v, x, y$. If Assumption 3.6 holds in addition to the conditions in Theorem 4.4, then (4.7) becomes $\check{\beta}_z - 1 \sim_d -\Delta(p_T/q_T)(P/(2S))$ for $z = v, x, y$ by Theorem 3.4. On the other hand, Assumption 4.2 holds for $r(z) = z$ if and only if $\mathbb{E}(V_t^2)$ and $\mathbb{E}(\sigma^2(V_t))$ are finite. Consequently, when $r(z) = z$, we have $(\check{\beta}_z - 1)/\Delta \not\to_p -\kappa$ for a linear drift diffusion (4.3) satisfying either $\mathbb{E}(V_t^2) = \infty$ or $\mathbb{E}(\sigma^2(V_t)) = \infty$. For instance, a GARCH diffusion satisfies $(\check{\beta}_z - 1)/\Delta \to_p -\kappa$ if $\mathbb{E}(V_t^2) < \infty$ with $\sigma^2 < 2\kappa$, whereas $(\check{\beta}_z - 1)/\Delta \to_p -\sigma^2/2$ if $\mathbb{E}(V_t^2) = \infty$ with $2\kappa < \sigma^2$.

As a conclusion of this section, let us remark that there is a large literature considering for autoregressions in discrete time models and allowing for heavy tails. In particular, Hill (2015) and Hill and Prokhorov (2016) propose a robust generalized empirical likelihood (GEL) method for estimation and inference of an autoregression that may have a heavy tailed heteroscedastic error. We expect that the GEL estimator can also be robust to fat tails in continuous time models. However, it is questionable whether the GEL estimator can be robust to the non-Markovianity of (x_i) and (y_i) in our framework. We leave the asymptotic properties of GEL methods in volatility regression for future research.

5. Extensions

5.1. *Asymptotic Negligibility of Errors in Realized Variance*

In Section 4, we analyze the asymptotic behaviors of the IV estimators under the assumption that each of V and D is independent of W^P. In reality, however, it is widely believed that there exist the leverage effect, which corresponds to a negative correlation between past returns and future volatility. As an extension of our previous results, we allow arbitrary dependences among V, D and W^P, and analyze the asymptotic negligibility of the errors in the realized variance.

Assumption 5.1. *(a) For ω in Assumption 3.1, $(\delta/\Delta^2)T(\omega^6)T\log^3(T/\delta) \to_p 0$, and (b) $\Delta T_D^2 \to_p 0$.*

It can be seen from the primary asymptotics in Proposition 4.1 (c) that the impact of errors in (y_i) may become asymptotically negligible as long as $\delta/\Delta^2 \to 0$ sufficiently quickly. Assumption 5.1 (a) is a sufficient condition for the asymptotic negligibility of the error, and requires more faster rate of convergence $\delta \to 0$ than Assumption 3.3. On the other hand, Assumption 5.1 (b) has a similar role to Assumption 3.4, and provides a sufficient condition for the asymptotic negligibility of the drift part (D_t) in the IV estimation with (y_i).

Proposition 5.1. *Under Assumptions 3.1-3.2, 4.1, and 5.1,*

$$\tilde{\beta}_y^{(k)} - 1 \sim_p \tilde{\beta}_x^{(k)} - 1 \quad \textit{and} \quad \check{\beta}_y^{(k)} - 1 \sim_p \check{\beta}_x^{(k)} - 1.$$

Unlike Proposition 4.1 (c), $\tilde{\beta}_y^{(k)}$ becomes asymptotically equivalent to $\tilde{\beta}_x^{(k)}$ regardless of the presence of leverage effects, especially, when $\delta/\Delta^2 \to 0$ sufficiently quickly. It is also true that, under the conditions in Proposition 5.1, $\hat{\beta}_y^{(k)} - 1 \sim_p \hat{\beta}_x^{(k)} - 1$ since $\hat{\beta}_z^{(k)}$ is a special case of $\tilde{\beta}_z^{(k)}$ with $r(z) = z$.

5.2. Fixed Δ Asymptotics for GARCH Diffusion

In Section 4.2, we obtained general asymptotics of the IV estimator $\check{\beta}_z$, which is robust to fat tails as well as errors in observed variance measures, under, in particular, the assumption that $\Delta \to 0$. In our asymptotics, the main motivation of introducing the small Δ assumption is to effectively handle general variance processes V having potentially unbounded moments. In practice, however, the volatility measure, (x_i) or (y_i), is often computed on a daily basis, and Δ is commonly fixed to a length of day. Under the fixed Δ, one may be interested in the quality of our approximation based on $\Delta \to 0$.

To see the usefulness of our asymptotics under $\Delta \to 0$, we consider a GARCH diffusion (2.7), and show that the fixed Δ asymptotics are approximately equivalent to the asymptotics obtained under $\Delta \to 0$. Specifically, we have

Theorem 5.2. *Let V be a GARCH diffusion (2.7) with $\mathbb{E}|V_t| < \infty$. If r is bounded, then $\check{\beta}_z \to_p \exp(-\kappa\Delta)$ as $T \to \infty$.*

Obviously, $\exp(-\kappa\Delta) = 1 - \kappa\Delta + o(\Delta)$, and the leading term $1 - \kappa\Delta$ is equivalent to the limit of $\check{\beta}_z$ obtained under $\Delta \to 0$ in Theorem 4.4. We also note again that our asymptotics of $\hat{\beta}_z$ in Section 3 provide the same results as those derived by Andersen et al. (2004) when the required moments are satisfied. Therefore, we may conclude that our asymptotics, obtained under $\Delta \to 0$, provide a useful asymptotic approximation at least for some popular models.

We heuristically consider the Cauchy estimator in Section 4.1. Our general theory under $\Delta \to 0$ is not directly applicable to the Cauchy estimator since the instrument $r(z - \overline{z}_N)$ is involving nondiffrentiable function $r(z) = \mathrm{sign}(z)$. Under a fixed Δ, however, we do not need the differentiability of the transformation r as long as we have proper conditional moments such as (2.10) and (2.11) for GARCH diffusions. In general, the required conditional moments are not available for most diffusion models having fat tails. This is another reason why our asymptotic approximation obtained under small Δ may be useful.

Given the asymptotic assumption of $\Delta \to 0$ as well as continuity of the sample path of V, two consecutive measures z_{i+1} and z_i are supposed to be very close for $z = v, x, y$. Consequently, we always have unit roots in $z = v, x, y$, and $\hat{\beta}_z, \tilde{\beta}_z, \check{\beta}_z \to_p 1$ as long as $\Delta \to 0$ sufficiently quickly as derived in the previous sections. Indeed, there are many evidences supporting the unit root like behavior in volatility regressions. In the empirical studies in Hansen and Lunde (2014), for instance, volatility regressions at daily frequency are considered for 29 assets in the Dow Jones industrial average. The range of parameter estimates for the coefficient of the first order autoregression with realized variances are $[0.611, 0.887]$ and $[0.895, 1.037]$, respectively, for the OLS and IV estimates. They also find that the volatility processes are highly persistent, and they fail to reject the unit root hypothesis at the 1% level for some volatility processes.

6. Simulations

In this section, we study by simulation the behavior of the Hill tail index estimator as well as the OLS and some IV estimators. For our simulations, we use the GARCH diffusion (2.7) with three sets of parameters. The first one is $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$ which implies that the corresponding V_t has a finite second moment since $\psi_0 = \sigma_0^2/(2\kappa_0) = 0.296 < 1$. This set of parameters was used by Andersen and Bollerslev (1998) as implied from the (weak) daily GARCH(1,1) model estimates for the DM/dollar from 1987 through 1992 using the temporal aggregation results of Drost and Nijman (1993) and Drost and Werker (1996); the same parameters were used by Andersen et al. (2004).

To consider a process with an unbounded variance of V_t, we consider two other sets of parameters by keeping the same κ_0 and μ_0, while we multiply σ_0^2 by 4 and 16, corresponding to $\psi_0 = 1.183$ and $\psi_0 = 4.732$, that is $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$. Clearly, the third model has thicker tails than the second one.

The simulation samples are generated by the Euler discretization at 10 seconds for $T = 250, 500, 1000$ days corresponding to 1, 2 and 4 years. We assume that the market is open 24 hours. For each day ($\Delta = 1$), we set the daily spot variance as the spot variance at the end of the day, while we compute the integrated variance by the numerical integration of the simulated spot variance process at 10 seconds. As for the realized variance, we analyze the frequency effects by considering three different frequencies: 10 minutes ($\delta/\Delta = 1/144$), 5 minutes ($\delta/\Delta = 1/288$) and 1 minute ($\delta/\Delta = 1/1440$). For each sample, we get rid of the first five days to reduce the effect of the initial value, and we do 10,000 replications.

6.1. *Tail Index*

We start by studying the properties of the Hill estimator by estimating the tail index of the returns, spot and integrated volatility of the GARCH diffusion model. In an important contribution, Nelson (1990) proved that when the length of the returns goes to zero, the returns follows (up to a scaling factor) a Student distribution with degree of freedom $\nu_0 = 2 + 4\kappa_0/\sigma_0^2 = 2 + 2/\psi_0$. Consequently, the tail index of the return is ν_0 which equals 8.75 for Model 1, 3.69 for Model 2 and 2.43 for Model 3.

Likewise, the stationary distribution of any stationary scalar diffusion process is well known and proportional to the speed density function $m(\cdot)$ defined in (2.6). One can easily show that $m(v) \sim v^{-2-1/\psi_0}$ when $v \to \infty$, implying that the tail index of the spot variance V_t equals $1 + 1/\psi_0$. Consequently, the tail index of the returns equals the double of the spot variance's tail index when the length of the returns goes to zero.

Unfortunately we do not know the tail indexes of the integrated and realized variances. There is no general result connecting the tail of a process with the tail of temporal aggregation version of it.

Figures 4 and 5 depict the average estimator of the tail index of the returns, the spot variance, the integrated variance and the three realized volatility measures of the three models. The averages are computed over 10,000 replications of samples with 1,000 observations each. The first panel of Figure 4 depicts the tail index of daily returns. If the Nelson's approximation is good, the true tail index should be 8.75 for Model 1, 3.69 for Model 2, and 2.43 for Model 3. The simulations suggest that there is negative bias in the Hill estimator, which is quite small for low values of k (we do not use the subscript n) and increases when k increases. However the order of the tails is coherent across models. The bias is maybe genuine, or because the Nelson's approximation is not good for our sample frequency. The second panel of Figure 4 depicts the tail index of the spot variance. The tail index should be 4.38 for Model 1, 1.85 for Model 2, and 1.22 for Model 3. There is clearly a positive bias when k is small and then the tail Hill estimator looks good when k increases. Again, the order of the tails is coherent across models with the right magnitudes. The third panel of Figure 4 depicts the tail index of the integrated variance for which we do not know the true tail index. The plots are quite similar to those of the spot variance. Figure 5 depicts the tail index of the three realized volatility measures for which we do not know the true index. The graphs are close to those of the integrated variance, especial for the third panel that corresponds to realized variance computed with 1-minute returns.

6.2. OLS and IV Estimations

We now turn to study of the empirical distributions of the OLS and IV estimators. We keep the three models of the GARCH diffusion (2.7), with three sample sizes, 250, 500, and 1,000.

We start by considering the regression

$$v_{i+1} = \alpha + \beta v_i + u_{i+1}, \quad \text{with } v_i = V_{i\Delta}.$$

We will focus on the slope parameter β. It is well known that β equals $\exp(-\kappa\Delta)$ when the spot variance has a finite second moment. However, we proved in (2.10) that the same result holds when V_t is stationary and has a finite first moment. Therefore, the slope of interest is $\exp(-\kappa\Delta)$ for the three models considered in this section.

When Δ is fixed and the second moment of V_t is bounded, the OLS estimator of β is consistent. Characterizing the fixed Δ asymptotics of the OLS is difficult when the second moment of V_t is not finite. However, we may deduce from Theorem 3.4

(see also Remark 3.3 (d)) that the OLS is inconsistent. However, the IV estimator is consistent by Theorem 5.2, when the Cauchy estimator is used.

Figure 6 depicts the empirical distribution of the OLS and IV estimators of the slope coefficient. The first panel deals with Model 1 for which the second moment is bounded, while the second and third panels deal respectively with Models 2 and 3 for which the second moment of V_t is unbounded. The figures are coherent with the theory. For Model 1 (top panel), both OLS and IV estimators look consistent with better properties when the sample size increases. However, the OLS estimator presents a bias and looks inconsistent, as expected by our theory, for Model 2 (second panel) and especially Model 3 (third panel) which present very fat tails. In contrast, the Cauchy estimator looks consistent for the two models, even though there is some bias that decreases when the sample sizes increases.

In practice, the spot variance process is not observed. It is therefore important to focus on feasible methods based on the observed realized variance processes. Accordingly, we consider the multi-period moment restriction (2.11) which is always valid for the spot and integrated variances, and is valid for the realized variance when the drift D_t is zero as in our simulations.[6] Consequently, we consider the moment condition

$$\mathbb{E}[r(z_{i-2})(z_i - \alpha - \beta z_{i-1})] = 0,$$

where z_i is either the spot, the integrated or one of the three realized variance measures. We consider two IV estimators: the first one is $r(z_{i-2}) = z_{i-2}$ while the second one is the sign of z_{i-2} minus its empirical mean, that is the Cauchy estimator. For the second and third models, the first IV estimator with $r(z_{i-2}) = z_{i-2}$ does not fulfill the restriction $\mathbb{E}[\| \, r(z_{i-2})(z_i - \alpha - \beta z_{i-1}) \, \|] < \infty$, and hence, we may deduce from (3.8) and (4.7) that it is not consistent for $\exp(-\kappa\Delta)$, even when $\Delta \to 0$. However, the corresponding estimator is consistent for the first model. The second IV estimator is the Cauchy one and leads to a consistent estimator for the three models.

Figures 7-11 depict respectively the empirical distribution of the slope's estimator for the five volatility measures listed above. Each figure contains three panels corresponding to the three models. For all figures, the Cauchy instrument based estimator

[6] The presence of a drift will introduce a small bias that will disappear when the length of the intra-day returns δ goes to zero.

looks consistent whether $\mathbb{E}(V_t^2) < \infty$ (first panel of each figure) or not (second and their panels of each figure), which is coherent with the theory. Importantly, the estimator based on observed volatility measures is consistent, which is practically more relevant. As expected, the IV estimator with $r(z_{i-2}) = z_{i-2}$ looks consistent only for Model 1 (first panel), but it looks inconsistent for Models 2 and 3, with a larger bias for Model 3.

7. Conclusion

Fat tails are a well-known empirical fact of financial returns. Surprisingly, the realized volatility literature ignored this fact. After proving empirically that the second moment of several realized variance measures are probably unbounded, we studied theoretically the limiting behavior of the OLS estimator of simple autoregressions of spot, integrated and realized variances. We proved that when the second moment of the spot variance is unbounded, the OLS estimators converge to random variables. Our theory is also valid when the second moment of the spot variance is bounded. In this case, the OLS estimates converge to finite and deterministic quantities which are the same ones derived by Andersen et al. (2004) in population regressions. Our theoretical results are based on asymptotic approximations. Both the simulations and the comparison with the results in Andersen et al. (2004) when the spot variance has a finite second moment corroborate the good quality of our approach.

In order to derive more positive results, we considered a GARCH diffusion process with unbounded second moment for the variance process and then we provided a consistent estimation method based on instrumental variable approach where the instrument is the sign of the lagged value of the variable of interest.

There is an important question that should be addressed. It concerns the forecast that one should compute under fat tails in a non-parametric setting. Various approaches could be considered like different loss functions or nonlinear transforms of the variable of interest. This question is currently under investigation.

Appendix

Throughout, we let f and σ^2 be twice continuously differentiable functions. Furthermore, we assume that for $g = \iota, f, f', f'', \mu, \sigma^2, \sigma^{2'}, \sigma^{2''}$, there is a locally bounded function ω such that $|g(v)| \leq \omega(v)$ for all $v \in \mathcal{D}$.

A. Useful Lemmas for Integrated Variance

Lemma A.1. *Let M be a continuous local martingale with its quadratic variation $[M]$ satisfying*

$$\sup_{0\leq t,s\leq T} \sup_{|t-s|\leq\delta} |[M]_t - [M]_s| = O_p\left(\delta\xi(\delta,T)\right)$$

for some sequence $(\xi(\delta,T))$ of positive numbers. As $\delta\xi(\delta,T) \to_p 0$, we have

$$\sup_{0\leq t,s\leq T} \sup_{|t-s|\leq\delta} |M_t - M_s| = O_p\left(\sqrt{\delta\xi(\delta,T)\log(T/\delta)}\right).$$

Proof of Lemma A.1. The stated result follows immediately from Theorem 1 and Corollary 1 of Kanaya et al. (2017). □

Lemma A.2. *If $\Delta^{1/2}T(\omega^2)\sqrt{\log(T/\Delta)} \to_p 0$, then*

$$\sup_{1\leq i\leq N} |f(x_i) - f(v_{i-i})| = O_p\left(\Delta T(f'\mu)\right) + O_p\left(\Delta^{1/2}T(f'\sigma)\sqrt{\log(T/\Delta)}\right) = o_p\left(1\right).$$

Proof of Lemma A.2. Since V has a continuous sample path, we may deduce from the mean value theorem that

$$\begin{aligned}\sup_{1\leq i\leq N} |f(x_i) - f(v_{i-i})| &= \sup_{1\leq i\leq N} \left| f'(V_{k_i})\frac{1}{\Delta}\int_{(i-1)\Delta}^{i\Delta}(V_t - V_{(i-1)\Delta})dt\right| \\ &\leq T(f') \sup_{1\leq i\leq N} \left|\frac{1}{\Delta}\int_{(i-1)\Delta}^{i\Delta}(V_t - V_{(i-1)\Delta})dt\right| \qquad \text{(A.1)}\end{aligned}$$

for some $k_i \in [(i-1)\Delta, i\Delta]$. Moreover, it follows from Lemma A.1 that

$$\sup_{0\leq t,s\leq T} \sup_{|t-s|\leq\Delta} |V_t - V_s| = O_p\left(\Delta T(\mu)\right) + O_p\left(\Delta^{1/2}T(\sigma)\sqrt{\log(T/\Delta)}\right), \qquad \text{(A.2)}$$

and hence,

$$\sup_{1\le i\le N}\left|\int_{(i-1)\Delta}^{i\Delta}(V_t - V_{(i-1)\Delta})dt\right| = O_p\left(\Delta^2 T(\mu)\right) + O_p\left(\Delta^{3/2}T(\sigma)\sqrt{\log(T/\Delta)}\right). \quad \text{(A.3)}$$

The stated result follows immediately from (A.1) and (A.3). □

Lemma A.3. *Under the condition in Lemma A.2, we have*

$$\Delta\sum_{i=1}^{N} f(x_i) = \int_0^T f(V_t)dt + O_p\left(\Delta^{1/2}T(\omega^2)T\sqrt{\log(T/\Delta)}\right).$$

Proof of Lemma A.3. Due to Lemma A.2, we have

$$\Delta\sum_{i=1}^{N}(f(x_i) - f(v_{i-1})) = O_p\left(\Delta T(f'\mu)T\right) + O_p\left(\Delta^{1/2}T(f'\sigma)T\sqrt{\log(T/\Delta)}\right). \quad \text{(A.4)}$$

Moreover, by Lemma B1 of Kim and Park (2017), we have

$$\begin{aligned}\Delta\sum_{i=1}^{N} f\left(v_{i-1}\right) = \int_0^T f(V_t)dt &+ O_p\left(\Delta T(f'\mu)T\right) + O_p\left(\Delta T(f''\sigma)T\right)\\ &+ O_p\left(\Delta T(f'\sigma)T^{1/2}\right), \quad \text{(A.5)}\end{aligned}$$

from which, together with (A.4), the stated result follows immediately. □

Lemma A.4. *If* $\Delta T(\omega^2)T \to_p 0$, *then*

$$\frac{1}{\Delta^2}\sum_{i=1}^{N}\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta - s)^2(f(V_s) - f(V_{i\Delta}))ds = o_p(1).$$

Proof of Lemma A.4. Due to Ito's lemma, we have

$$\frac{1}{\Delta^2}\sum_{i=1}^{N}\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta - s)^2(f(V_s) - f(V_{i\Delta}))ds = A_T + B_T, \quad \text{(A.6)}$$

where

$$A_T = \frac{1}{\Delta^2}\sum_{i=1}^{N}\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta - s)^2\left(\int_{i\Delta}^{s}(f'\mu + f''\sigma^2/2)(V_t)dt\right)ds,$$

$$B_T = \frac{1}{\Delta^2}\sum_{i=1}^{N}\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta - s)^2\left(\int_{i\Delta}^{s}(f'\sigma)(V_t)dW_t\right)ds.$$

For A_T, it is easy to see that

$$A_T = O_p\left(\Delta T(f'\mu)T\right) + O_p\left(\Delta T(f''\sigma^2)T\right). \tag{A.7}$$

As for B_T, we have

$$\begin{aligned} B_T &= \frac{1}{\Delta^2}\sum_{i=1}^{N}\int_{i\Delta}^{(i+1)\Delta}(f'\sigma)(V_t)\left(\int_{t}^{(i+1)\Delta}((i+1)\Delta - s)^2 ds\right)dW_t \\ &= O_p\left(\Delta T(f'\sigma)T^{1/2}\right), \end{aligned} \tag{A.8}$$

where the first line is due to the changing the order of integrals, and the second line can be deduced from the proof of Lemma B1 in Kim and Park (2017). The stated result follows immediately from (A.6)-(A.8).

□

Lemma A.5. *If $\Delta^{1/2}T(\omega^{5/2})T\sqrt{\log(T/\Delta)} \to_p 0$, we have*

$$\sum_{i=1}^{N} f(v_{i-1})(x_{i+1} - x_i)^2 = \frac{2}{3}\int_0^T f(V_t)\sigma^2(V_t)dt + o_p(1).$$

Proof of Lemma A.5. We write

$$\begin{aligned} \sum_{i=1}^{N} f(v_{i-1})(x_{i+1} - x_i)^2 &= \frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{i\Delta}^{(i+1)\Delta}(V_t - V_{i\Delta})dt + \int_{(i-1)\Delta}^{i\Delta}(V_{i\Delta} - V_t)dt\right)^2 \\ &= A_T + B_T + R_T, \end{aligned} \tag{A.9}$$

where

$$A_T = \frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{i\Delta}^{(i+1)\Delta}(V_t - V_{i\Delta})dt\right)^2,$$
$$B_T = \frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{(i-1)\Delta}^{i\Delta}(V_{i\Delta} - V_t)dt\right)^2,$$
$$R_T = \frac{2}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{i\Delta}^{(i+1)\Delta}(V_t - V_{i\Delta})dt\right)\left(\int_{(i-1)\Delta}^{i\Delta}(V_{i\Delta} - V_t)dt\right).$$

Due to (A.9), the stated result follows immediately if we show

$$A_T, B_T = \frac{1}{3}\int_0^T f(V_t)\sigma^2(V_t)dt + O_p\left(\Delta^{1/2}T(\omega^{5/2})T\sqrt{\log(T/\Delta)}\right), \tag{A.10}$$
$$R_T = O_p\left(\Delta^{1/2}T(\omega^{5/2})T\sqrt{\log(T/\Delta)}\right). \tag{A.11}$$

Proof of (A.10). We will only prove the result for A_T, since the proof of the result for B_T is entirely analogous. For the proof, we write A_T as

$$\begin{aligned} A_T &= \frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\mu(V_s)ds dt + \int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\sigma(V_s)dW_s dt\right)^2 \\ &= A_{1,T} + A_{2,T} + A_{3,T}, \end{aligned} \tag{A.12}$$

where

$$A_{1,T} = \frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\mu(V_s)ds dt\right)^2,$$
$$A_{2,T} = \frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\sigma(V_s)dW_s dt\right)^2,$$
$$A_{3,T} = \frac{2}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\mu(V_s)ds dt\right)\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\sigma(V_s)dW_s dt\right).$$

For $A_{1,T}$, we have

$$|A_{1,T}| \leq \sup_{0\leq t\leq T} |(f\mu^2)(V_t)|\Delta T = O_p\left(\Delta T(f\mu^2)T\right). \tag{A.13}$$

On the other hand, it follows from Lemma A.1 that $A_{3,T}$ satisfies

$$\begin{aligned} |A_{3,T}| &\leq 2T(f\mu)T \sup_{0\leq t,s\leq T} \sup_{|t-s|\leq\Delta} \left|\int_s^t \sigma(V_u)dW_u\right| \\ &= O_p\left(\Delta^{1/2}T(f\mu\sigma)T\sqrt{\log(T/\Delta)}\right). \end{aligned} \tag{A.14}$$

As for $A_{2,T}$, we define a continuous martingale M as

$$M_t = \sum_{i=1}^{j-1} \int_{i\Delta}^{(i+1)\Delta} ((i+1)\Delta - s)\sigma(V_s)dW_s + \int_{j\Delta}^t ((j+1)\Delta - s)\sigma(V_s)dW_s$$

for $t \in [j\Delta, (j+1)\Delta)$, $j = 1, 2, \cdots, N-1$, so that we have

$$\begin{aligned} A_{2,T} &= \frac{1}{\Delta^2}\sum_{i=1}^N f(V_{(i-1)\Delta})\left(M_{(i+1)\Delta} - M_{i\Delta}\right)^2 \\ &= \frac{1}{\Delta^2}\sum_{i=1}^N f(V_{(i-1)\Delta})\int_{i\Delta}^{(i+1)\Delta} d[M]_t + \frac{2}{\Delta^2}\sum_{i=1}^N f(V_{(i-1)\Delta})\int_{i\Delta}^{(i+1)\Delta}(M_t - M_{i\Delta})dM_t, \end{aligned} \tag{A.15}$$

where the last line follows from Ito's lemma.

For the second term of (A.15), we can deduced from Lemma B5 of Kim and Park (2017) that

$$\begin{aligned} &\frac{2}{\Delta^2}\sum_{i=1}^N f(V_{(i-1)\Delta})\int_{i\Delta}^{(i+1)\Delta}(M_t - M_{i\Delta})dM_t \\ &= \frac{2}{\Delta^2}\sum_{i=1}^N f(V_{(i-1)\Delta})\int_{i\Delta}^{(i+1)\Delta}\left(\int_{i\Delta}^t ((i+1)\Delta - s)\sigma(V_s)dW_s\right)((i+1)\Delta) - t)\sigma(V_t)dW_t \\ &= O_p\left(\Delta^{1/2}T(f\sigma^2)T^{1/2}\sqrt{\log(T/\Delta)}\right). \end{aligned} \tag{A.16}$$

For the first term of (A.15), we have

$$
\begin{aligned}
\frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\int_{i\Delta}^{(i+1)\Delta} d[M]_t &= \frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta-s)^2\sigma^2(V_s)ds \\
&= \frac{1}{\Delta^2}\sum_{i=1}^{N}(f\sigma^2)(V_{(i-1)\Delta})\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta-s)^2 ds + S_T \\
&= \frac{1}{3}\sum_{i=1}^{N}(f\sigma^2)(V_{(i-1)\Delta})\Delta + S_T, \qquad (A.17)
\end{aligned}
$$

where

$$
\begin{aligned}
S_T &= \frac{1}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta-s)^2(\sigma^2(V_s)-\sigma^2(V_{(i-1)\Delta}))ds \\
&= O_p\left(\Delta T(f\sigma^{2'}\mu)T\right) + O_p\left(\Delta T(f\sigma^{2''}\sigma^2)T\right) + O_p\left(\Delta T(f\sigma^{2'}\sigma)T^{1/2}\right) \qquad (A.18)
\end{aligned}
$$

by Lemma A.4. Moreover, it follows from (A.5) that

$$
\begin{aligned}
\sum_{i=1}^{N}(f\sigma^2)(V_{(i-1)\Delta})\Delta = \int_0^T (f\sigma^2)(V_t)dt &+ O_p\left(\Delta T((f\sigma^2)'\mu)T\right) + O_p\left(\Delta T((f\sigma^2)''\sigma^2)T\right) \\
&+ O_p\left(\Delta T((f\sigma^2)'\sigma)T^{1/2}\right),
\end{aligned}
$$

from which, toghether with (A.16)-(A.18) and the conditions in the lemma, we have

$$
A_{2,T} = \frac{1}{3}\int_0^T (f\sigma^2)(V_t)dt + O_p\left(\Delta^{1/2}T(\omega^2)T^{1/2}\sqrt{\log(T/\Delta)}\right). \qquad (A.19)
$$

Therefore, we can obtain (A.10) by applying (A.13), (A.14) and (A.19) to (A.12).

Proof of (A.11). We write

$$
R_T = R_{1,T} + R_{2,T} + R_{3,T} + R_{4,T}, \qquad (A.20)
$$

where

$$R_{1,T} = \frac{2}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{t}\mu(V_s)dsdt\right)\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\mu(V_s)dsdt\right),$$
$$R_{2,T} = \frac{2}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{t}\sigma(V_s)dW_sdt\right)\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\mu(V_s)dsdt\right),$$
$$R_{3,T} = \frac{2}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{t}\mu(V_s)dsdt\right)\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\sigma(V_s)dW_sdt\right),$$
$$R_{4,T} = \frac{2}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{t}\sigma(V_s)dW_sdt\right)\left(\int_{i\Delta}^{(i+1)\Delta}\int_{i\Delta}^{t}\sigma(V_s)dW_sdt\right).$$

We can easily show that

$$R_{1,T} = O_p\left(\Delta T(f\mu^2)T\right). \tag{A.21}$$

Similarly as in (A.14), we have

$$R_{2,T}, R_{3,T} = O_p\left(\Delta^{1/2}T(f\mu\sigma)T\sqrt{\log(T/\Delta)}\right). \tag{A.22}$$

By changing the order of integrals, we rewrite $R_{4,T}$ as

$$R_{4,T} = \frac{2}{\Delta^2}\sum_{i=1}^{N} f(V_{(i-1)\Delta})\left(\int_{(i-1)\Delta}^{i\Delta}(i\Delta - s)\sigma(V_s)dW_s\right)\left(\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta - s)\sigma(V_s)dW_s\right)$$

and define a continuous martingale M as

$$\begin{aligned}M_t =& \frac{2}{\Delta^2}\sum_{i=1}^{j-1} f(V_{(i-1)\Delta})\left(\int_{(i-1)\Delta}^{i\Delta}(i\Delta - s)\sigma(V_s)dW_s\right)\left(\int_{i\Delta}^{(i+1)\Delta}((i+1)\Delta - s)\sigma(V_s)dW_s\right)\\ &+ \frac{2}{\Delta^2}f(V_{(j-1)\Delta})\left(\int_{(j-1)\Delta}^{j\Delta}(j\Delta - s)\sigma(V_s)dW_s\right)\left(\int_{j\Delta}^{t}((j+1)\Delta - s)\sigma(V_s)dW_s\right)\end{aligned}$$

for $t \in [j\Delta, (j+1)\Delta)$, $j = 1, 2, \cdots, N-1$, so that we have $M_T = R_{4,T}$. Then we have

$$\begin{aligned}[M]_T &= \frac{4}{\Delta^4} \sum_{i=1}^{N} f^2(V_{(i-1)\Delta}) \left(\int_{(i-1)\Delta}^{i\Delta} (i\Delta - s)\sigma(V_s) dW_s \right)^2 \left(\int_{i\Delta}^{(i+1)\Delta} ((i+1)\Delta - s)^2 \sigma^2(V_s) ds \right) \\ &= O_p\left(\Delta T (f^2\sigma^4) T \log(T/\Delta)\right)\end{aligned}$$

since

$$\sup_{1 \leq i \leq N} \left(\int_{i\Delta}^{(i+1)\Delta} ((i+1)\Delta - s)^2 \sigma^2(V_s) ds \right) = O_p\left(\Delta^3 T(\sigma^2)\right)$$

and

$$\sum_{i=1}^{N} \left(\int_{(i-1)\Delta}^{i\Delta} (i\Delta - s)\sigma(V_s) dW_s \right)^2 = O_p\left(\Delta^2 T(\sigma^2) T \log(T/\Delta)\right),$$

similarly as in (A.14). Therefore, we have $R_{4,T} = O_p\left(\Delta^{1/2} T (f\sigma^2) T^{1/2} \sqrt{\log(T/\Delta)}\right)$, from which, together with (A.20)-(A.22), we have (A.11). □

Lemma A.6. *Under the condition in Lemma A.5, we have*

$$\sum_{i=k+1}^{N} f(V_{i-k-1})(V_{i\Delta} - V_{(i-1)\Delta})(V_{(i-k)\Delta} - V_{(i-k-1)\Delta}) = o_p(1)$$

for any positive integer $k \geq 1$.

Proof of Lemma A.6. We have

$$\sum_{i=k+1}^{N} f(V_{i-k-1})(V_{i\Delta} - V_{(i-1)\Delta})(V_{(i-k)\Delta} - V_{(i-k-1)\Delta}) = A_T + B_T + C_T + D_T, \quad \text{(A.23)}$$

where

$$A_T = \sum_{i=j+1}^{N} f(V_{i-k-1}) \left(\int_{(i-1)\Delta}^{i\Delta} \mu(V_s) ds \right) \left(\int_{(i-k-1)\Delta}^{(i-k)\Delta} \mu(V_s) ds \right),$$

$$B_T = \sum_{i=j+1}^{N} f(V_{i-k-1}) \left(\int_{(i-1)\Delta}^{i\Delta} \sigma(V_s) dW_s \right) \left(\int_{(i-k-1)\Delta}^{(i-k)\Delta} \mu(V_s) ds \right),$$

$$C_T = \sum_{i=j+1}^{N} f(V_{i-k-1}) \left(\int_{(i-1)\Delta}^{i\Delta} \mu(V_s) ds \right) \left(\int_{(i-k-1)\Delta}^{(i-k)\Delta} \sigma(V_s) dW_s \right),$$

$$D_T = \sum_{i=j+1}^{N} f(V_{i-k-1}) \left(\int_{(i-1)\Delta}^{i\Delta} \sigma(V_s) dW_s \right) \left(\int_{(i-k-1)\Delta}^{(i-k)\Delta} \sigma(V_s) dW_s \right).$$

For A_T, we have

$$A_T = O_p \left(\Delta T (f\mu^2) T \right). \tag{A.24}$$

Moreover, we have

$$B_T, C_T = O_p \left(\Delta^{1/2} T (f\mu\sigma) T \sqrt{\log(T/\Delta)} \right). \tag{A.25}$$

similarly as in (A.14).

As for D_T, we may show that

$$D_T = O_p \left(\Delta^{1/2} T (f\sigma^2) T^{1/2} \sqrt{\log(T/\Delta)} \right) \tag{A.26}$$

similarly as in the proof for $R_{4,T}$ in (A.20). The stated result follows immediately from (A.23)-(A.26). □

Lemma A.7. *Let the condition in Lemma A.5 hold. Then for $k \geq 0$ we have*

$$\sum_{i=k+1}^{N} f(v_{i-k-1})(x_{i+1} - x_{i-k})^2 = \left(\frac{2}{3} + k \right) \int_0^T f(V_t) \sigma^2(V_t) dt + o_p(1).$$

Proof of Lemma A.7. Since we have

$$x_{i+1} - x_{i-k} = \int_{i\Delta}^{(i+1)\Delta} (V_t - V_{i\Delta})dt + \int_{(i-k-1)\Delta}^{(i-k)\Delta} (V_{(i-k)\Delta} - V_t)dt + \Delta(V_{i\Delta} - V_{(i-k)\Delta}),$$

we may write

$$\sum_{i=k+1}^{N} f(v_{i-k-1})(x_{i+1} - x_{i-k})^2 = A_T + B_T + C_T + R_{1,T} + R_{2,T} + R_{3,T}, \tag{A.27}$$

where

$$\begin{aligned}
A_T &= \frac{1}{\Delta^2} \sum_{i=k+1}^{N} f(V_{(i-k-1)\Delta}) \left(\int_{i\Delta}^{(i+1)\Delta} (V_t - V_{i\Delta})dt \right)^2, \\
B_T &= \frac{1}{\Delta^2} \sum_{i=k+1}^{N} f(V_{(i-k-1)\Delta}) \left(\int_{(i-k-1)\Delta}^{(i-k)\Delta} (V_{(i-k)\Delta} - V_t)dt \right)^2, \\
C_T &= \sum_{i=k+1}^{N} f(V_{(i-k-1)\Delta})(V_{i\Delta} - V_{(i-k)\Delta})^2, \\
R_{1,T} &= \frac{2}{\Delta^2} \sum_{i=k+1}^{N} f(V_{(i-k-1)\Delta}) \left(\int_{i\Delta}^{(i+1)\Delta} (V_t - V_{i\Delta})dt \right) \left(\int_{(i-k-1)\Delta}^{(i-k)\Delta} (V_{(i-k)\Delta} - V_t) \right), \\
R_{2,T} &= \frac{2}{\Delta} \sum_{i=k+1}^{N} f(V_{(i-k-1)\Delta}) \left(\int_{i\Delta}^{(i+1)\Delta} (V_t - V_{i\Delta})dt \right) (V_{i\Delta} - V_{(i-k)\Delta}), \\
R_{3,T} &= \frac{2}{\Delta} \sum_{i=k+1}^{N} f(V_{(i-k-1)\Delta})(V_{i\Delta} - V_{(i-k)\Delta}) \left(\int_{(i-k-1)\Delta}^{(i-k)\Delta} (V_{(i-k)\Delta} - V_t) \right).
\end{aligned}$$

Similarly as in the proofs of (A.10) and (A.11) in Lemma A.5, we may show that

$$A_T, B_T = \frac{1}{3} \int_0^T (f\sigma^2)(V_t)dt + O_p\left(\Delta^{1/2} T(\omega^{5/2}) T \sqrt{\log(T/\Delta)} \right) \tag{A.28}$$

$$R_{1,T}, R_{2,T}, R_{3,T} = O_p\left(\Delta^{1/2} T(\omega^{5/2}) T \sqrt{\log(T/\Delta)} \right). \tag{A.29}$$

As for C_T, we have

$$C_T = \sum_{j=0}^{k-1} \sum_{i=k+1}^{N} f(V_{i-k-1})(V_{(i-j)\Delta} - V_{(i-j-1)\Delta})^2 + o_p(1)$$
$$= k \int_0^T (f\sigma^2)(V_t)dt + o_p(1), \tag{A.30}$$

where the first equality is due to Lemma A.6, and the last equality can be deduced from the proof of Lemma A.5 with Lemma A11 of Kim and Park (2018). The stated result is then follows from (A.27)-(A.30). □

Lemma A.8. *Let the condition in Lemma A.5 hold. Then for $k \geq 0$ we have*

$$\sum_{i=k+1}^{N} f(x_{i-k}^*)(x_{i+1} - x_{i-k})^2 = \sum_{i=k+1}^{N} f(v_{i-k-1})(x_{i+1} - x_{i-k})^2 + o_p(1).$$

Proof of Lemma A.8. We have

$$\sum_{i=k+1}^{N} |f(x_{i-k}^*) - f(v_{i-k-1})|(x_{i+1} - x_{i-k})^2 \leq \sum_{i=k+1}^{N} (x_{i+1} - x_{i-k})^2 \left(\sup_{1\leq i\leq N-k} |f(x_i^*) - f(v_{i-1})| \right)$$
$$\leq \left((2/3 + k) \int_0^T \sigma^2(V_t)dt \right) \left(\sup_{0\leq s,t\leq T} \sup_{|s-t|\leq k\Delta} |f(V_s) - f(V_t)| \right) (1 + o_p(1))$$
$$= O_p(T(\sigma^2)T) \times \left(O_p\left(\Delta T(f'\mu + f''\sigma^2/2)\right) + O_p\left(\Delta^{1/2}T(f'\sigma)\sqrt{\log(T/\Delta)}\right) \right)$$
$$= o_p(1),$$

where the second line follows from Lemma A.7 with the construction of (x_i^*), the third line can be deduced from (A.2) with Ito's lemma, the last line follows from the condition in this lemma. □

B. Useful Lemmas for Realized Variance

In the following, we write $e_i = y_i - x_i$ and $e_i = e_i^a + 2e_i^b + e_i^c$ with

$$\begin{aligned}
e_i^a &= \frac{1}{\Delta}\sum_{j=1}^{n}\left(\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} D_t dt\right)^2, \\
e_i^b &= \frac{1}{\Delta}\sum_{j=1}^{n}\left(\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} D_t dt\right)\left(\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} V_t^{1/2} dW_t^P\right), \\
e_i^c &= \frac{1}{\Delta}\sum_{j=1}^{n}\left(\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} V_t^{1/2} dW_t^P\right)^2 - \frac{1}{\Delta}\int_{(i-1)\Delta}^{i\Delta} V_t dt \\
&= \frac{2}{\Delta}\sum_{j=1}^{n}\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta}\left(\int_{(i-1)\Delta+(j-1)\delta}^{t} V_s^{1/2} dW_s^P\right) V_t^{1/2} dW_t^P,
\end{aligned}$$

where, in particular, the last equality follows from Ito's lemma.

B.1. With Leverage Effects

In this section, we assume that V, D and W^P are arbitrarily dependent.

Lemma B.1. *If (a)* $(\delta/\Delta)T(\omega^2)\log(T/\delta) \to_p 0$ *and (b)* $\Delta T_D^2 \to_p 0$, *then*

$$\sup_{1\le i\le N} |y_i - x_i| = O_p\left((\delta/\Delta)^{1/2} T(\iota)\log(T/\delta)\right).$$

Proof of Lemma B.1. We have

$$\sup_{1\le i\le N} |e_i^a| = O_p\left(\delta T_D^2\right) \quad \text{and} \quad \sup_{1\le i\le N} |e_i^b| = O_p\left(\sqrt{\delta T_D^2 T(\iota)\log(T/\delta)}\right) \tag{B.1}$$

since, in particular, we have

$$\sup_{\delta\le t\le T}\left|\int_{t-\delta}^{t} V_u^{1/2} dW_u^P\right| = O_p\left(\sqrt{\delta T(\iota)\log(T/\delta)}\right) \tag{B.2}$$

by Lemma A.1 with the condition (a). It follows that

$$\begin{aligned}\sup_{1\le i\le N}|y_i-x_i| &\le \sup_{1\le i\le N}|e_i^a| + \sup_{1\le i\le N}|e_i^b| + \sup_{1\le i\le N}|e_i^c| \\ &= \sup_{1\le i\le N}|e_i^c| + o_p\left(\sqrt{(\delta/\Delta)T(\iota)\log(T/\delta)}\right) \qquad \text{(B.3)}\end{aligned}$$

due to the conditions (a) and (b).

As for $\sup_{1\le i\le N}|e_i^c|$ in (B.3), we define a continuous martingale M as $M_0=0$ and

$$M_t - M_{(k-1)\delta} = \frac{2}{\Delta}\int_{(k-1)\delta}^{t}\left(\int_{(k-1)\delta}^{s} V_u^{1/2}dW_u^P\right)V_s^{1/2}dW_s^P \qquad \text{(B.4)}$$

for $t\in[(k-1)\delta,k\delta)$, $k=1,2,\cdots,nN$ so that $M_{i\Delta}-M_{(i-1)\Delta}=e_i^c$ for $i=1,2,\cdots,N$. The quadratic variation $[M]$ of M satisfies

$$\begin{aligned}\sup_{(i-1)\Delta\le t\le i\Delta}|[M]_t-[M]_{(i-1)\Delta}| &= \frac{4}{\Delta^2}\sum_{j=1}^{n}\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta}\left(\int_{(i-1)\Delta+(j-1)\delta}^{t}V_s^{1/2}dW_s^P\right)^2 V_t dt \\ &= O_p\left((\delta/\Delta)T(\iota^2)\log(T/\delta)\right)\end{aligned}$$

uniformly in $1\le i\le N$, due to (B.2). We then use Lemma A.1 with the condition (a) so that we have

$$\sup_{1\le i\le N}|e_i^c| \le \sup_{1\le i\le N}\sup_{(i-1)\Delta\le t\le i\Delta}|M_t-M_{(i-1)\Delta}| = O_p\left(\sqrt{(\delta/\Delta)T(\iota^2)\log^2(T/\delta)}\right) \qquad \text{(B.5)}$$

from which, jointly with (B.3), we have the stated result. □

Lemma B.2. *Under the conditions in Lemma B.1, we have*

$$\begin{aligned}\sup_{1\le i\le N}|f(y_i)-f(x_i)| &= O_p\left((\delta/\Delta)^{1/2}T(\omega^2)\log(T/\delta)\right), \\ \Delta\sum_{i=1}^{N}f(y_i) &= \Delta\sum_{i=1}^{N}f(x_i) + O_p\left((\delta/\Delta)^{1/2}T(\omega^2)T\log(T/\delta)\right).\end{aligned}$$

Proof of Lemma B.2. Since f is differentiable, we have

$$\sup_{1\le i\le N} |f(y_i) - f(x_i)| \le T(f') \sup_{1\le i\le N} |y_i - x_i| = O_p\left((\delta/\Delta)^{1/2} T(f'\iota)\log(T/\delta)\right),$$

from which we have the stated results. □

Lemma B.3. *Under the conditions in Lemma B.1, we have*

$$(a)\ \sum_{i=1}^{N} f(v_{i-1})(e_i)^2, \quad \sum_{i=1}^{N-1} f(v_{i-1})e_i e_{i+1} = O_p\left((\delta/\Delta^2)T(\omega^3)T\log^2(T/\delta)\right),$$
$$(b)\ \sum_{i=k+1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)(e_{i+1}-e_i) = O_p\left((\delta^{1/2}/\Delta)T(\omega^3)T\log^{3/2}(T/\delta)\right).$$

Proof of Lemma B.3. It follows from Lemma B.1 that

$$\begin{aligned}\sum_{i=1}^{N} f(v_{i-1})(e_i)^2, \quad \sum_{i=1}^{N-1} f(v_{i-1})\,|e_i e_{i+1}| &\le \frac{T}{\Delta} \sup_{1\le i\le N} (e_i)^2 \sup_{1\le i\le N} |f(v_{i-1})| \\ &= O_p\left((\delta/\Delta^2)T(f\iota^2)T\log^2(T/\delta)\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=k+1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)(e_{i+1}-e_i) &\le 2\frac{T}{\Delta} \sup_{1\le i\le N} |e_i| \sup_{\Delta\le t\le T} |V_t - V_{t-\Delta}| \sup_{1\le i\le N} |f(v_{i-1})| \\ &= O_p\left(\sqrt{(\delta/\Delta^2)T(f^2\omega^4)T^2\log^3(T/\delta)}\right)\end{aligned}$$

by (A.2). □

Lemma B.4. *Let the conditions in Lemma B.1 hold. If* $(\delta/\Delta^2)T(\omega^6)T\log^3(T/\delta) \to_p 0$*, then for any* $k \ge 0$*, we have*

$$\sum_{i=k+1}^{N-1} f(v_{i-k-1})(y_{i+1}-y_{i-k})^2 = \sum_{i=k+1}^{N-1} f(v_{i-k-1})(x_{i+1}-x_{i-k})^2 + o_p(T^{1/2}).$$

Proof of Lemma B.4. This follows immediately from Lemma B.3. □

Lemma B.5. *Let the conditions in Lemmas A.2 and B.1 hold. Then, for any $k \geq 0$, we have*

$$\sum_{i=k+1}^{N-1} |f(y_{i-k}^*) - f(v_{i-k-1})|(y_{i+1} - y_{i-k})^2$$
$$= \sum_{i=k+1}^{N-1} (y_{i+1} - y_{i-k})^2 \times \left(O_p\left((\delta/\Delta)^{1/2} T(\omega^2) \log(T/\delta)\right) + O_p\left(\Delta^{1/2} T(\omega^2) \sqrt{\log(T/\Delta)}\right)\right).$$

Proof of Lemma B.5. We have

$$\sum_{i=k+1}^{N-1} |f(y_{i-k}^*) - f(v_{i-k-1})|(y_{i+1} - y_{i-k})^2 \leq \sum_{i=k+1}^{N-1} (y_{i+1} - y_{i-k})^2 \left(A_N + B_N\right),$$

where $A_N = \sup_{1 \leq i \leq N-k} |f(y_i^*) - f(y_i)|$, and

$$\begin{aligned} B_N &= \sup_{1 \leq i \leq N} |f(y_i) - f(v_{i-1})| \leq \sup_{1 \leq i \leq N} |f(y_i) - f(x_i)| + \sup_{1 \leq i \leq N} |f(x_i) - f(v_{i-1})| \\ &= O_p\left((\delta/\Delta)^{1/2} T(\omega^2) \log(T/\delta)\right) + O_p\left(\Delta^{1/2} T(\omega^2) \sqrt{\log(T/\Delta)}\right) \end{aligned}$$

by Lemmas A.2 and B.2.

Since $y_i^* \in [y_{i-k}, y_{i+1}]$, we have

$$\begin{aligned} A_N &\leq \sup_{1 \leq i,j \leq N} \sup_{|i-j| \leq k} |f(y_i) - f(y_j)| \\ &\leq \sup_{1 \leq i,j \leq N} \sup_{|i-j| \leq k} |f(v_i) - f(v_j)| + 2 \sup_{1 \leq i \leq N} |f(y_i) - f(v_{i-1})| \\ &\leq T(f') \times \sup_{1 \leq i,j \leq N} \sup_{|i-j| \leq k} |V_{i\Delta} - V_{j\Delta}| + 2 \sup_{1 \leq i \leq N} |f(y_i) - f(v_{i-1})| \\ &= O_p\left(\Delta^{1/2} T(f'\omega) \sqrt{\log(T/\Delta)}\right) + 2 \sup_{1 \leq i \leq N} |f(y_i) - f(v_{i-1})|, \end{aligned}$$

where the last line follows from (A.2). □

B.2. Without Leverage Effects

Now we let each of V and D is independent of W^P.

Lemma B.6. *Let the condition (a) in Lemma B.1 hold. If* $(\delta/\Delta)T_D^4 T \to_p 0$, *then*

$$\begin{aligned}
(a)\quad & \sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_{i+1}^a,\ \sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_i^a \\
& = O_p\left(\sqrt{\delta T(\omega^4)T\log(T/\Delta)}\right), \\
(b)\quad & \sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_{i+1}^b,\ \sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_i^b \\
& = O_p\left(\sqrt{(\delta^3/\Delta)^{1/2}T(\omega^5)T^{1/2}\log(T/\Delta)}\right), \\
(c)\quad & \sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_i^c,\ \sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_{i+1}^c \\
& = O_p\left(\sqrt{(\delta/\Delta)T(\omega^6)T\log(T/\Delta)\log(T/\delta)}\right).
\end{aligned}$$

Proof of Lemma B.6 (a). We have

$$\begin{aligned}
& \sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_{i+1}^a,\ \sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_i^a \\
& \leq \frac{2}{\Delta}\left(\sup_{1\leq i\leq N}|f(v_{i-1})e_i^a|\right)\sum_{i=1}^{N-1}\int_{i\Delta}^{(i+1)\Delta}|V_s - V_{s-\Delta}|ds \\
& = O_p\left((\delta/\Delta^{1/2})T_D^2 T(f\omega)T\sqrt{\log(T/\Delta)}\right)
\end{aligned}$$

by (A.2) and (B.1). The stated result is then follows immediately under the conditions in the lemma. □

Proof of Lemma B.6 (b). We will only prove the result for $\sum_{i=1}^{N-1}(x_{i+1}-x_i)e_i^b$, since the proofs of the results for $\sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_i^b$ and $\sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_{i+1}^b$ with a locally bounded f are entirely analogous.

We define a continuous process M as $M_0 = 0$ and

$$\begin{aligned}
& M_t - M_{(i-1)\Delta+(j-1)\delta} \\
& = \left(\int_{i\Delta}^{(i+1)\Delta}(V_s - V_{s-\Delta})ds\right)\left(\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} D_s ds\right)\left(\int_{(i-1)\Delta+(j-1)\delta}^{t} V_s^{1/2}dW_s^P\right)
\end{aligned}$$

for $t \in [(i-1)\Delta + (j-1)\delta, (i-1)\Delta + j\delta)$ with $i = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, n$. Since each of D and W is independent of W^P, M_t becomes a continuous martingale with respect to the filteration $\mathcal{G}_t = \mathcal{F}_t^{W^P} \vee \mathcal{F}_\infty^W \vee \mathcal{F}_\infty^D$, where $(\mathcal{F}_t^Z)$ is the natural filteration of $Z = W^P, W, D$. The quadratic variation $[M]$ satisfies

$$\begin{aligned}[M]_T =& \sum_{i=1}^{N-1} \left(\int_{i\Delta}^{(i+1)\Delta} (V_s - V_{s-\Delta}) ds \right)^2 \\ & \times \left(\sum_{j=1}^{n} \left(\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} D_u du \right)^2 \left(\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} V_s ds \right) \right) \\ =& O_p\left(\delta^2 \Delta^3 T_D^2 T(\omega^3) T \log(T/\Delta) \right)\end{aligned}$$

by (A.2). The stated result for $\sum_{i=1}^{N-1}(x_{i+1}-x_i)e_i^b$ follows immediately since $\sum_{i=1}^{N-1}(x_{i+1}-x_i)e_{i+1}^b = (1/\Delta^2)M_T$, which completes the proof. □

Proof of Lemma B.6 (c). We will only prove the result for $\sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_i^c$, since the proof of the result for $\sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1}-x_i)e_{i+1}^c$ is entirely analogous.

We define a continuous process M as $M_0 = 0$ and

$$\begin{aligned}M_t - M_{(i-1)\Delta+(j-1)\delta} =& f(V_{(i-1)\Delta}) \left(\int_{i\Delta}^{(i+1)\Delta} (V_s - V_{s-\Delta}) ds \right) \\ & \times \left(\int_{(i-1)\Delta+(j-1)\delta}^{t} \left(\int_{(i-1)\Delta+(j-1)\delta}^{s} V_u^{1/2} dW_u^P \right) V_s^{1/2} dW_s^P \right)\end{aligned}$$

for $t \in [(i-1)\Delta + (j-1)\delta, (i-1)\Delta + j\delta)$ with $i = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, n$. Since W^P is independent of W, M_t becomes a continuous martingale in a similar argument to the proof of Lemma B.6 (b). The quadratic variation $[M]$ of M satisfies

$$\begin{aligned}[M]_T =& \sum_{i=1}^{N-1} f^2(V_{(i-1)\Delta}) \left(\int_{i\Delta}^{(i+1)\Delta} (V_s - V_{s-\Delta}) ds \right)^2 \\ & \times \left(\sum_{j=1}^{n} \int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} \left(\int_{(i-1)\Delta+(j-1)\delta}^{s} V_u^{1/2} dW_u^P \right)^2 V_s ds \right) \\ =& O_p\left(\delta \Delta^3 T(f^2\omega^4) T \log(T/\Delta) \log(T/\delta) \right)\end{aligned}$$

due to (A.2) and (B.2). The stated result follows immediately since $\sum_{i=1}^{N-1} f(v_{i-1})(x_{i+1} - x_i)e_i^c = (2/\Delta^2)M_T$, which completes the proof. $\square$

Lemma B.7. *If* $(\delta/\Delta)T(\omega^2)T\log(T/\delta) \to 0$, *then*

$$\sum_{i=1}^{N} f(v_{i-1})(e_i^c)^2 = \frac{2\delta}{\Delta^2}\int_0^T f(V_t)V_t^2 dt + O_p\left((\delta^2/\Delta^3)^{1/2}T(\omega^3)T^{1/2}\log(T/\delta)\right).$$

Proof of Lemma B.7. Let M be the martingale defined in (B.4). Then, by Ito's lemma, we have

$$\begin{aligned}\sum_{i=1}^{N} f(v_{i-1})(e_i^c)^2 &= \sum_{i=1}^{N} f(v_{i-1})(M_{i\Delta} - M_{(i-1)\Delta})^2 \\ &= \sum_{i=1}^{N} f(v_{i-1})\int_{(i-1)\Delta}^{i\Delta} d[M]_t + 2\sum_{i=1}^{N} f(v_{i-1})\int_{(i-1)\Delta}^{i\Delta}(M_t - M_{(i-1)\Delta})dM_t.\end{aligned}$$

To complete the proof, it suffice to show that

$$\sum_{i=1}^{N} f(v_{i-1})\int_{(i-1)\Delta}^{i\Delta} d[M]_t = \frac{2\delta}{\Delta^2}\int_0^T f(V_t)V_t^2 dt + O_p\left((\delta^{3/2}/\Delta^2)T(f\omega^2)T\log^{3/2}(T/\delta)\right), \tag{B.6}$$

$$\sum_{i=1}^{N} f(v_{i-1})\int_{(i-1)\Delta}^{i\Delta}(M_t - M_{(i-1)\Delta})dM_t = O_p\left((\delta/\Delta^{3/2})T(f\omega^2)T^{1/2}\log(T/\delta)\right). \tag{B.7}$$

since $\delta T\log(T/\delta) = o(\Delta)$.

Below we prove (B.6) and (B.7) separately when $f(v) = 1$ for all $v \in \mathcal{D}$ since the proof of the results for a locally bounded f is entirely analogous.

Proof of (B.6). Due to Ito's lemma, we have

$$[M]_T = \frac{4}{\Delta^2}\sum_{k=1}^{nN}\int_{(k-1)\delta}^{k\delta}\left(\int_{(k-1)\delta}^{t} V_s^{1/2}dW_s^P\right)^2 V_t dt = \frac{2\delta}{\Delta^2}\int_0^T V_t^2 dt + A_T + 2B_T + 2C_T,$$

where

$$A_T = \frac{4}{\Delta^2}\sum_{k=1}^{nN}\int_{(k-1)\delta}^{k\delta}\left(\int_{(k-1)\delta}^{t} V_s ds\right) V_t dt - \int_0^T V_t^2 dt,$$

$$B_T = \frac{4}{\Delta^2}\sum_{k=1}^{nN}\int_{(k-1)\delta}^{k\delta}\left(\int_{(k-1)\delta}^{t}\left(\int_{(k-1)\delta}^{s} V_u^{1/2} dW_u^P\right) V_s^{1/2} dW_s^P\right)(V_t - V_{(k-1)\delta})dt,$$

$$C_T = \frac{4}{\Delta^2}\sum_{k=1}^{nN} V_{(k-1)\delta}\int_{(k-1)\delta}^{i\delta}\left(\int_{(k-1)\delta}^{t}\left(\int_{(k-1)\delta}^{s} V_u^{1/2} dW_u^P\right) V_s^{1/2} dW_s^P\right) dt.$$

For A_T, we write $A_T = A_{1T} + A_{2T} - (1/2)A_{3T}$, where

$$A_{1T} = \frac{4}{\Delta^2}\sum_{k=1}^{K}\int_{(k-1)\delta}^{k\delta}\left(\int_{(k-1)\delta}^{t}(V_s - V_{(k-1)\delta})ds\right) V_t dt,$$

$$A_{2T} = \frac{4}{\Delta^2}\sum_{k=1}^{K} V_{(k-1)\delta}\int_{(k-1)\delta}^{k\delta}\left(\int_{(k-1)\delta}^{t} ds\right)(V_t - V_{(k-1)\delta})dt,$$

$$A_{3T} = \frac{2\delta}{\Delta^2}\sum_{k=1}^{K}\int_{(k-1)\delta}^{k\delta}(V_t^2 - V_{(k-1)\delta}^2)dt = \frac{2\delta}{\Delta^2}\sum_{k=1}^{K}\int_{(k-1)\delta}^{k\delta}(V_t + V_{(k-1)\delta})(V_t - V_{(k-1)\delta})dt.$$

By (A.2), we have

$$A_{1T}, A_{2T}, A_{3T} = O_p\left((\delta/\Delta)^2 T(\mu\iota)T\right) + O_p\left((\delta^{3/2}/\Delta^2)T(\sigma\iota)T\sqrt{\log(T/\delta)}\right). \quad \text{(B.8)}$$

As for B_T, we successively apply Lemma A.1 to have

$$\sup_{1\le k\le nN}\ \sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}\left(\int_{(k-1)\delta}^{s} V_u^{1/2} dW_u^P\right) V_s^{1/2} dW_s^P\right| = O_p\left(\delta T(\iota)\log(T/\delta)\right),$$

from which, jointly with (A.2), we have

$$B_T = O_p\left((\delta/\Delta)^2 T(\mu\iota)T\log(T/\delta)\right) + O_p\left((\delta^{3/2}/\Delta^2)T(\sigma\iota)T\log^{3/2}(T/\delta)\right). \quad \text{(B.9)}$$

Similar as in the proof of B.6 (c), we may show that

$$C_T = \frac{4}{\Delta^2}\sum_{k=1}^{K} V_{(k-1)\delta}\int_{(k-1)\delta}^{k\delta}(k\delta - s)\left(\int_{(k-1)\delta}^{s} V_u^{1/2}dW_u^P\right)V_s^{1/2}dW_s^P$$
$$= O_p\left((\delta^{3/2}/\Delta^2)T(\iota^2)\sqrt{T\log(T/\delta)}\right). \qquad \text{(B.10)}$$

The stated result (B.6) follows immediately from (B.8)-(B.10).

Proof of (B.7). We define a continuous martingale $\tilde{M}$ as $\tilde{M} = 0$ and

$$\tilde{M}_t - \tilde{M}_{(i-1)\Delta}$$
$$= \sum_{k=1}^{j-1}\int_{(i-1)\Delta+(k-1)\delta}^{(i-1)\Delta+k\delta}(M_s - M_{(i-1)\Delta})dM_s + \int_{(i-1)\Delta+(j-1)\delta}^{t}(M_s - M_{(i-1)\Delta})dM_s$$

for $t \in [(i-1)\Delta + (j-1)\delta, (i-1)\Delta + j\delta)$ with $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, n$, so that

$$\tilde{M}_T = \sum_{i=1}^{N}\int_{(i-1)\Delta}^{i\Delta}(M_t - M_{(i-1)\Delta})dM_t.$$

The quadratic variation $[\tilde{M}]$ of $\tilde{M}$ satisfies

$$[\tilde{M}]_T = \sum_{i=1}^{N}\sum_{j=1}^{n}\int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta}(M_t - M_{(i-1)\Delta})^2 d[M]_t$$
$$\leq [M]_T\left(\sup_{1\leq i\leq N}\ \sup_{(i-1)\Delta\leq t\leq i\Delta}(M_t - M_{(i-1)\Delta})^2\right)$$
$$= O_p\left((\delta/\Delta^2)T(\iota^2)T\right) \times O_p\left((\delta/\Delta)T(\iota^2)\log^2(T/\delta)\right)$$

where the last equality follows from (B.6) and (B.5). This completes the proof of (B.7). □

Lemma B.8. *Under the condition (a) in Lemma B.1, we have*

$$\sum_{i=1}^{N-1} f(v_{i-1})e_i^c e_{i+1}^c = O_p\left(\sqrt{(\delta^2/\Delta^3)T(\omega^6)T\log^3(T/\delta)}\right)$$

Proof of Lemma B.8. We define a continuous martingale M as $M_t = 0$ for $0 \leq t < \Delta$, and

$$M_t - M_{i\Delta+(j-1)\delta} = \frac{2}{\Delta}\left(f(v_{i-1})e_i^c\right)\left(\int_{i\Delta+(j-1)\delta}^t \left(\int_{i\Delta+(j-1)\delta}^s V_u^{1/2} dW_u^P\right) V_s^{1/2} dW_s^P\right)$$

for $t \in [i\Delta + (j-1)\delta, i\Delta + j\delta)$ with $i = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, n$, so that $M_T = \sum_{i=1}^{N-1} e_i^c e_{i+1}^c$. Then, the quadratic variation $[M]$ of M satisfies

$$\begin{aligned}
[M]_T =& \frac{4}{\Delta^2}\sum_{i=1}^{N-1}\left(f(v_{i-1})e_i^c\right)^2\left(\sum_{j=1}^{n}\int_{i\Delta+(j-1)\delta}^{i\Delta+j\delta}\left(\int_{i\Delta+(j-1)\delta}^s V_u^{1/2} dW_u^P\right)^2 V_t dt\right)\\
\leq& \left(\sup_{1\leq i\leq N}\left(f(v_{i-1})e_i^c\right)^2\right)\left(\frac{4}{\Delta^2}\sum_{i=1}^{nN}\int_{(i-1)\delta}^{i\delta}\left(\int_{(i-1)\delta}^s V_u^{1/2} dW_u^P\right)^2 V_t dt\right)\\
=& O_p\left((\delta^2/\Delta^3)T(f^2\iota^4)T\log^3(T/\delta)\right)
\end{aligned}$$

by (B.2) and (B.5). This completes the proof. □

Lemma B.9. *Under the conditions in Lemma B.6, we have*

$$\begin{aligned}
&\sum_{i=1}^{N-1} f(v_{i-1})(e_i^a)^2, \sum_{i=1}^{N-1} f(v_{i-1})e_i^a e_{i+1}^a = O_p\left(\delta T(\omega)\right),\\
&\sum_{i=1}^{N-1} f(v_{i-1})(e_i^b)^2, \sum_{i=1}^{N-1} f(v_{i-1})e_i^b e_{i+1}^b = O_p\left((\delta/\Delta)^{1/2} T(\omega^2) T^{1/2}\log(T/\delta)\right),
\end{aligned}$$

and

$$\sum_{i=1}^{N-1} f(v_{i-1})e_i^a e_i^b, \sum_{i=1}^{N-1} f(v_{i-1})e_i^a e_{i+1}^b, \sum_{i=1}^{N-1} f(v_{i-1})e_i^b e_{i+1}^a = O_p\left((\delta^3/\Delta^4)^{1/2} T(\omega^2)\sqrt{\log(T/\delta)}\right).$$

Proof of Lemma B.9. The stated results follow immediately from (B.1) with the conditions in the lemma. □

Lemma B.10. *Under the conditions in Lemma B.6, we have*

$$\begin{aligned}
&\sum_{i=1}^{N-1} f(v_{i-1})e_i^a e_i^c, \sum_{i=1}^{N-1} f(v_{i-1})e_i^a e_{i+1}^c, \sum_{i=1}^{N-1} f(v_{i-1})e_i^c e_{i+1}^a\\
&= O_p\left((\delta/\Delta)T(\omega^2)T^{1/2}\log(T/\delta)\right).
\end{aligned}$$

Proof of Lemma B.10. The stated result follows from (B.1) and (B.5). □

Lemma B.11. *Under the conditions in Lemma B.6, we have*

$$\sum_{i=1}^{N} f(v_{i-1})e_i^b e_i^c = O_p\left(\sqrt{(\delta^2/\Delta^3)T(\omega^3)T\log(T/\delta)}\right) + O_p\left((\delta/\Delta)T(\omega^{3/2})\log(T/\delta)\right).$$

Proof of Lemma B.11. We will only prove the result for $f(v) = 1$ since the result for a locally bounded f can be obtained similarly.

We define two continuous time processes M^b and M^c, as $M^c = M$, where M is defined in (B.4), and

$$M_t^b - M_{(k-1)\delta}^b = \frac{1}{\Delta}\left(\int_{(k-1)\delta}^{k\delta} D_t dt\right)\left(\int_{(k-1)\delta}^{t} V_s^{1/2} dW_s^P\right)$$

for $t \in [(k-1)\delta, k\delta)$ with $M_0^b = 0$. Note that M^b and M^c are continuous martingales since, in particular, D is independent of W^P. Moreover, $M_{i\Delta}^b - M_{(i-1)\Delta}^b = e_i^b$ and $M_{i\Delta}^c - M_{(i-1)\Delta}^c = e_i^c$. Then, by Ito's lemma, we have

$$\sum_{i=1}^{N} e_i^b e_i^c = \sum_{i=1}^{N}(M_{i\Delta}^b - M_{(i-1)\Delta}^b)(M_{i\Delta}^c - M_{(i-1)\Delta}^c) = A_T + B_T + C_T,$$

where

$$\begin{aligned}
A_T &= \frac{1}{\Delta^2}\sum_{k=1}^{nN}\left(\int_{(k-1)\delta}^{k\delta} D_t dt\right)\left(\int_{(k-1)\delta}^{k\delta}\left(\int_{(k-1)\delta}^{t} V_s^{1/2} dW_s^P\right) V_t dt\right),\\
B_T &= \frac{2}{\Delta^2}\sum_{i=1}^{N}\int_{(i-1)\Delta}^{i\Delta}(M_t^c - M_{(i-1)\Delta}^c)dM_t^b,\\
C_T &= \frac{2}{\Delta^2}\sum_{i=1}^{N}\int_{(i-1)\Delta}^{i\Delta}(M_t^b - M_{(i-1)\Delta}^b)dM_t^c.
\end{aligned}$$

By Lemma A.1, we have

$$\begin{aligned}
A_T &= O_p\left((\delta^{3/2}/\Delta^2)T_D T(\iota^{3/2})T\sqrt{\log(T/\delta)}\right)\\
&= O_p\left(\sqrt{(\delta^2/\Delta^3)T(\omega^3)T\log(T/\delta)}\right) \qquad \text{(B.11)}
\end{aligned}$$

due to the condition $(\delta/\Delta)T_D^4 T \to_p 0$ in this lemma.

For B_T, we define a continuous martingale $\tilde{M}^b$ as $\tilde{M}_0^b = 0$ and

$$\tilde{M}_t^b - \tilde{M}_{(i-1)\Delta}^b = \int_{(i-1)\Delta}^t (M_s^c - M_{(i-1)\Delta}^c) dM_s^b.$$

for $t \in [(i-1)\Delta, i\Delta)$ with $i = 1, 2, \cdots, N$, so that $\tilde{M}_T^b = B_T$. Then the quadratic variation $[\tilde{M}^b]$ of $\tilde{M}^b$ satisfies

$$\begin{aligned}
[\tilde{M}^b]_T &= \sum_{i=1}^N \int_{(i-1)\Delta}^{i\Delta} (M_t^c - M_{(i-1)\Delta}^c)^2 d[M^b]_t, \\
&\leq [M^b]_T \left(\sup_{1 \leq i \leq N} \sup_{(i-1)\Delta \leq t \leq i\Delta} (M_t^c - M_{(i-1)\Delta}^c)^2 \right) \\
&= O_p\left((\delta/\Delta)^2 T_D^2 T(\iota) T \right) \times O_p\left((\delta/\Delta) T(\iota^2) \log^2(T/\delta) \right),
\end{aligned} \tag{B.12}$$

where the last line follows from (B.5) with the construction of M^b. Similarly, we may define a continuous martingale $\tilde{M}^c$ such that $\tilde{M}_T^c = C_T$ and

$$\begin{aligned}
[\tilde{M}^c]_T &= \sum_{i=1}^N \int_{(i-1)\Delta}^{i\Delta} (M_t^b - M_{(i-1)\Delta}^b)^2 d[M^c]_t \\
&= O_p\left((\delta^2/\Delta) T_D^2 T(\iota) \log(T/\delta) \right) \times O_p\left((\delta/\Delta^2) T(\iota^2) T \log(T/\delta) \right).
\end{aligned} \tag{B.13}$$

It then follows from (B.12) and (B.13) that

$$\begin{aligned}
B_T, C_T &= O_p\left((\delta/\Delta)^{3/2} T_D T(\iota^{3/2}) T^{1/2} \log(T/\delta) \right) \\
&= O_p\left((\delta/\Delta) T(\omega^{3/2}) \log(T/\delta) \right)
\end{aligned} \tag{B.14}$$

by the condition $(\delta/\Delta)T_D^4 T \to_p 0$ in this lemma. The stated result follows from (B.11) and (B.14). $\square$

Lemma B.12. *Under the conditions in Lemma B.6, we have*

$$\sum_{i=1}^{N-1} f(v_{i-1}) e_{i+1}^b e_i^c, \; \sum_{i=1}^{N-1} f(v_{i-1}) e_i^b e_{i+1}^c = O_p\left((\delta/\Delta) T(\omega^{3/2}) \log(T/\delta) \right).$$

Proof of Lemma B.12. We will only prove the result for $\sum_{i=1}^{N-1} f(v_{i-1})e^b_{i+1}e^c_i$, since the proof of the result for $\sum_{i=1}^{N-1} f(v_{i-1})e^b_i e^c_{i+1}$ is entirely analogous.

Let M^b and M^c be the martingales defined in the proof of Lemma B.11. We define a continuous martingale $\tilde{M}^b$ as $\tilde{M}^b_t = 0$ for $0 \leq t < \Delta$, and

$$\tilde{M}^b_t - \tilde{M}^b_{i\Delta} = f(V_{(i-1)\Delta})(M^c_{i\Delta} - M^c_{(i-1)\Delta})(M^b_t - M^b_{i\Delta})$$

for $i = 1, 2, \cdots, N-1$, so that $\sum_{i=1}^{N-1} f(v_{i-1})e^b_{i+1}e^c_i = \tilde{M}^b_T$. Then the quadratic variation $[\tilde{M}^b]$ of $\tilde{M}^b$ satisfies

$$\begin{aligned}
[\tilde{M}^b]_T &= \sum_{i=1}^{N-1} f^2(V_{(i-1)\Delta})(M^c_{i\Delta} - M^c_{(i-1)\Delta})^2 \int_{i\Delta}^{(i+1)\Delta} d[M^b]_t, \\
&= O_p\big((\delta/\Delta)^2 T_D^2 T(\iota)T\big) \times O_p\big((\delta/\Delta)T(\iota^2)\log^2(T/\delta)\big) \times O_p(T(f^2))
\end{aligned}$$

similar as in (B.12), from which we have the stated result. $\square$

Lemma B.13. *Let the conditions in Lemmas A.5 and B.6 hold. If* $\delta/\Delta^2 = O(1)$ *and* $(\delta/\Delta)T(\omega^6)T\log^3(T/\delta) \to_p 0$, *then for any* $k \geq 0$, *we have*

$$\sum_{i=k+1}^{N-1} f(v_{i-k-1})(y_{i+1} - y_{i-k})^2 = \left(\frac{2}{3} + k\right)\int_0^T f(V_t)\sigma^2(V_t)dt + \frac{4\delta}{\Delta^2}\int_0^T f(V_t)V_t^2 dt + o_p(1).$$

Proof of Lemma B.13. We have

$$\begin{aligned}
&\sum_{i=k+1}^{N-1} f(v_{i-k-1})(y_{i+1} - y_{i-k})^2 - \sum_{i=k+1}^{N-1} f(v_{i-k-1})(x_{i+1} - x_{i-k})^2 \\
&= \sum_{i=k+1}^{N-1} f(v_{i-k-1})(e_{i+1} - e_{i-k})^2 + o_p(1) \\
&= \sum_{i=k+1}^{N-1} f(v_{i-k-1})(e^c_{i+1} - e^c_{i-k})^2 + o_p(1) \\
&= \sum_{i=k+1}^{N-1} f(v_{i-k-1})(e^c_{i+1})^2 + \sum_{i=k+1}^{N-1} f(v_{i-k-1})(e^c_{i-k})^2 + o_p(1)
\end{aligned}$$

where the first equality follows from Lemma B.6, the second equality holds due to

Lemmas B.9-B.12, and the last equality follows from Lemma B.8. Then the stated result can be deduced from Lemmas A.5 and B.7. □

Lemma B.14. *Let the conditions in Lemma B.13 hold. Then we have*

$$\sum_{i=k+1}^{N-1} |f(y_{i-k}^*) - f(v_{i-k-1})|(y_{i+1} - y_{i-k})^2$$
$$= O_p\left((\delta/\Delta)^{1/2} T(\omega^4) T \log(T/\delta)\right) + O_p\left(\Delta^{1/2} T(\omega^4) T \sqrt{\log(T/\Delta)}\right).$$

Proof of Lemma B.14. The stated result follows immediately from Lemmas B.5 and B.13. □

C. Proofs of Main Results

Proof of Lemma 2.1. The stochastic differential equation (2.7) has a solution

$$V_t = V_0 e^{\sigma W_t - (\kappa + \sigma^2/2)t} + \kappa\mu \int_0^t e^{\sigma(W_t - W_s) - (\kappa+\sigma^2/2)(t-s)} ds.$$

Since V_t is an homogeneous Markov process, we have

$$\mathbb{E}[V_{t+\Delta}|V_t] = f_\Delta(V_t), \quad \text{with} \quad f_\Delta(V_0) = \mathbb{E}[V_\Delta|V_0].$$

Note that f_Δ is well defined since the distribution of W_t is exponentially decaying. Moreover, it follows from $\mathbb{E}[\exp(cW_t)] = \exp(u^2 t/2)$ for all $c \in \mathbb{R}$ that

$$f_\Delta(V_0) = \mu + \exp(-\kappa\Delta)(V_0 - \mu),$$

from which we have the desired result. □

Proof of Proposition 2.2 (a). Define $U_t(\Delta) = (V_{t+\Delta} - \mu) - \exp(-\kappa\Delta)(V_t - \mu)$ for $t, \Delta > 0$. It then follows from Lemma 2.1 that

$$\mathbb{E}(U_t(\Delta)|\mathcal{F}_s^W) = \mathbb{E}(U_t(\Delta)|V_s) = 0 \quad \text{for all} \quad s \leq t, \tag{C.1}$$

where $(\mathcal{F}_t^W)$ is the natural filteration of (W_t), since V_t is an homogeneous Markov process. This completes the proof of the result for $z = v$.

As for the case $z = x$, we define $u_{i+1} = \frac{1}{\Delta}\int_{(i-1)\Delta}^{i\Delta} U_t(\Delta)dt$ for $i = 1, 2, \cdots, N$ so that $u_{i+1} = (x_{i+1} - \mu) - \exp(-\kappa\Delta)(x_i - \mu)$. It then follows from (C.1) that

$$\mathbb{E}(u_{i+1}|\mathcal{F}^W_{(i-1)\Delta}) = \frac{1}{\Delta}\int_{(i-1)\Delta}^{i\Delta} \mathbb{E}(U_t(\Delta)|\mathcal{F}^W_{(i-1)\Delta})dt = 0,$$

from which we have the result for $z = x$ since the sigma-field generated by $x_{i-1} = (1/\Delta)\int_{(i-2)\Delta}^{(i-1)\Delta} V_t dt$ is a subset of $\mathcal{F}^W_{(i-1)\Delta}$ for all $i = 1, 2, \cdots$. □

Proof of Proposition 2.2 (b). Under the conditions in the proposition, we have $y_i - x_i = e_i$, where

$$e_i = \int_{(i-1)\Delta+(j-1)\delta}^{(i-1)\Delta+j\delta} \left(\int_{(i-1)\Delta+(j-1)\delta}^{t} V_s^{1/2} dW_s^P \right) V_t^{1/2} dW_t^P$$

Let $(\mathcal{F}_t)$ be a filteration such that both (W_t) and (W_t^P) are adapted. Clearly, (e_i) is a m.d.s. with respect to $(\mathcal{F}_{i\Delta})$ satisfying $\mathbb{E}(e_i|\mathcal{F}_{j\Delta}) = 0$ for all $j \leq i-1$. Therefore, we have

$$\mathbb{E}(e_{i+1}|y_{i-1}) = \mathbb{E}(e_i|y_{i-1}) = 0$$

since the sigma-field generated by y_{i-1} is a subset of $\mathcal{F}_{(i-1)\Delta}$ for all $i = 1, 2, \cdots$. The stated result is then follows immediately from Part (a) of this proposition. □

Proof of Corollary 2.3. The stated result follows from Proposition 2.2. □

Proofs of Lemma 3.1 (a) and Proposition 4.1 (a). The stated results for (v_i) and (x_i) can be deduced from Lemmas A.2-A.3 and (A.2) with Assumptions 3.1-3.2. On the other hand, the results for (y_i) can be deduced from Lemmas A.2-A.3, (A.2) and Lemma B.2 with Assumptions 3.1-3.5. □

Proofs of Lemma 3.1 (b) and Proposition 4.1 (b). The stated results for (v_i) and (x_i) can be deduced from Lemmas A.5, A.7 and A.8 with Assumptions 3.1-3.2. The result for (y_i) can be obtained immediately from the result for (x_i) with Lemmas B.13-B.14 under Assumptions 3.1-3.5. □

Proofs of Proposition 3.2 and Proposition 4.1 (c). Due to Ito's lemma, Proposition 3.2 follows immediately from Lemma 3.1. Similarly, we can obtain Proposition 4.1 (c) from Proposition 4.1 (a)-(b). □

Proofs of Lemma 3.3, Theorem 3.4 and Theorem 4.2. The asymptotics in Theorem 3.4 can be obtained by applying Lemma 3.3 to the continuous time approximations in Proposition 3.2. Moreover, Lemma 3.3 follows immediately from Lemma 3.2 of Kim and Park (2018). Similarly, we can obtain Theorem 4.2 from Proposition 4.1 and Ito's lemma under, in particular, Assumption 4.2. □

Proof of Corollary 4.3. It can be shown that the scale density s' of the linear drift diffusion (4.3) defined on $(0, \infty)$ satisfies $s'(v) \to \infty$ as $v \to \infty$ under the stationarity and the condition $\sigma^2(v)/v^2 = O(1)$ as $v \to \infty$. Moreover, a stationary diffusion defined on $(0, \infty)$ also satisfies $s'(v) \to \infty$ as $v \to 0$ due to Lemma A6 of Kim and Park (2018).

For any $v_l, v_u \in (0, \infty)$, we have

$$-2 \int_{v_l}^{v_u} (m\mu r)(v) dv = \int_{v_u}^{v_l} (m\sigma^2 r')(v) dv - [(r/s')(v_u) - (r/s')(v_l)]$$

by the integration by parts, and therefore, we have

$$-2 \int_0^\infty (m\mu r)(v) dv = \int_0^\infty (m\sigma^2 r')(v) dv$$

since r is bounded, and $s'(v) \to \infty$ as $v \to v_B$ for $v_B = 0, \infty$. This completes the proof since $\pi(v) = m(v) / \int_{\mathcal{D}} m(v) dv$. □

Proof of Theorem 4.4. Due to (4.6), the stated results follow immediately if we show

$$\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k} - \overline{z}_N)\Delta = \sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k-1} - \overline{z}_N)\Delta + o_p(1). \tag{C.2}$$

But it follows from Taylor expansion that

$$\begin{aligned}
&\sum_{i=k+2}^{N-1} r(z_{i-k-1})(z_{i-k} - z_{i-k-1})\Delta \\
&= -\frac{\Delta}{2} \sum_{i=k+2}^{N-1} r'(z^*_{i-k-1})(z_{i-k} - z_{i-k-1})^2 + \Delta(r_1(z_{N-k-1}) - r_1(z_1)) \\
&= O_p\left(\Delta T(\omega^3)\right) + O_p\left(\Delta\right)
\end{aligned}$$

where the last equality follows from Proposition 4.1 (b) and the stationarity of V. Therefore, we have (C.2) under Assumption 3.2, from which, jointly with Proposition 4.1, we have the desired result. □

Proof of Proposition 5.1. It follows from Lemmas A.7, B.4 and B.5 with Assumptions 3.2, 4.1 and 5.1 that

$$\sum_{i=k+1}^{N-1} r'(y^*_{i-k})(y_{i+1} - y_{i-k})^2 = \sum_{i=k+1}^{N-1} r'(v_{i-k-1})(x_{i+1} - x_{i-k})^2 \left(1 + o_p(1)\right),$$

from which, together with Lemmas A.8 and B.2, we have the stated result. □

Proof of Theorem 5.2. This follows immediately from Corollary 2.3. □

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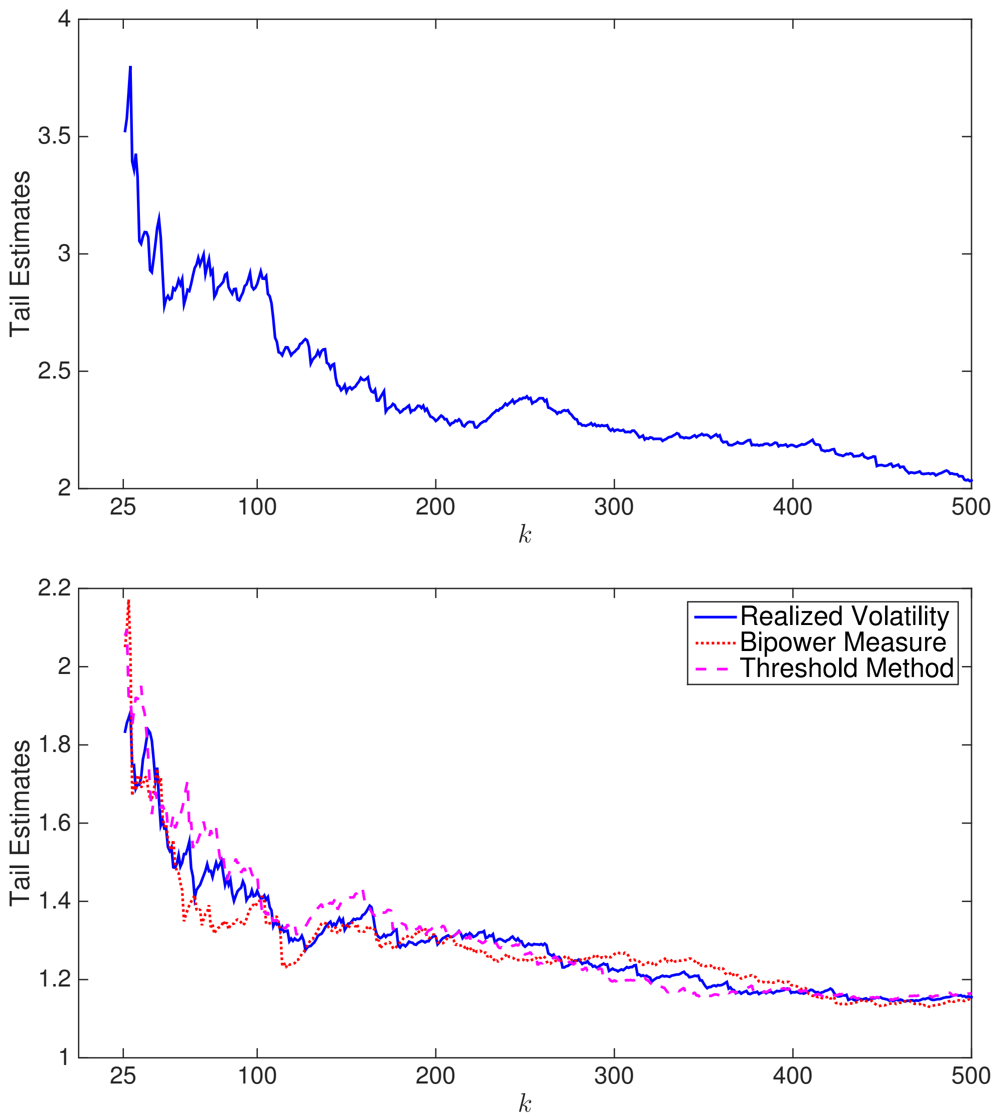
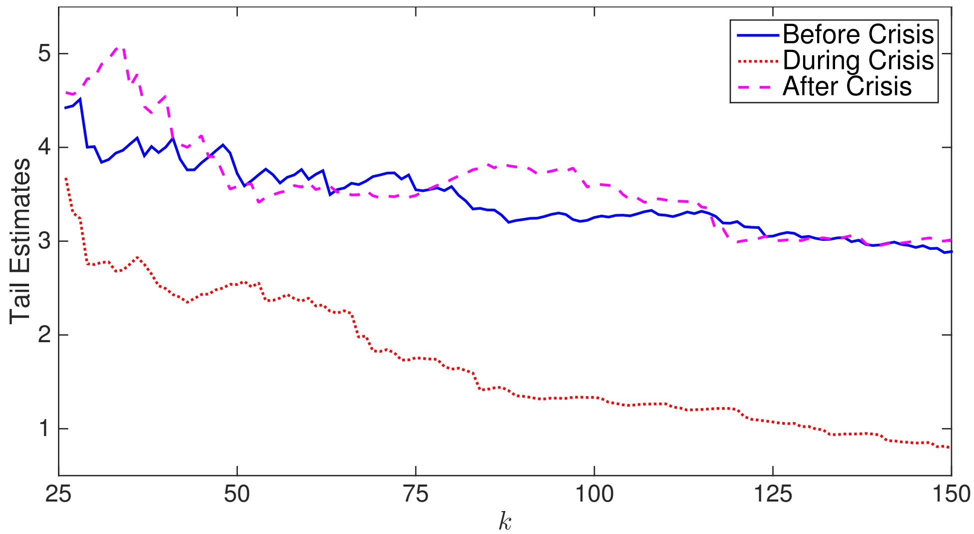


Fig. 1. **Tail index of returns and volatility measures.** The figures depict the Hill estimator of the tail index of the SPDR S&P 500 ETF (SPY), from June 15, 2004 through June 13, 2014. The first panel depicts the tail index of the daily return (open-to-close). The second panel depicts the tail index of the realized volatility, the bipower variation and the threshold volatility measure.



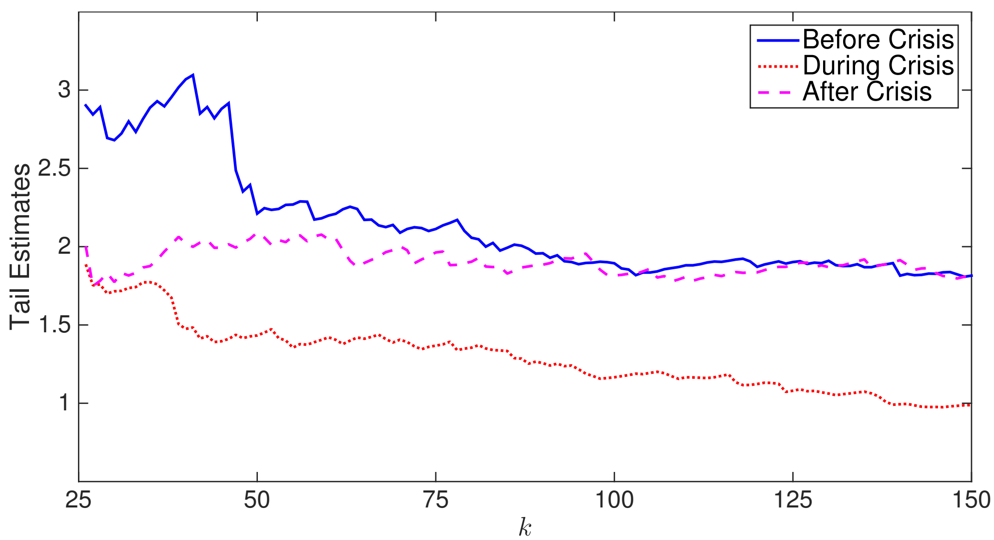


Fig. 2. **Tail index of returns and realized volatility over sub-periods.** The figures depict the Hill estimator of the tail index of the SPDR S&P 500 ETF (SPY). The full period is divided in three sub-periods: Before Crisis (June 15, 2004 through August 29, 2008), During Crisis (September 2, 2008 through May 29, 2009) and After Crisis (June 1, 2009 through June 13, 2014). The first panel depicts the tail index of the daily return (open-to-close) for the three periods. The second panel depicts the tail index of the realized volatility for the three periods.

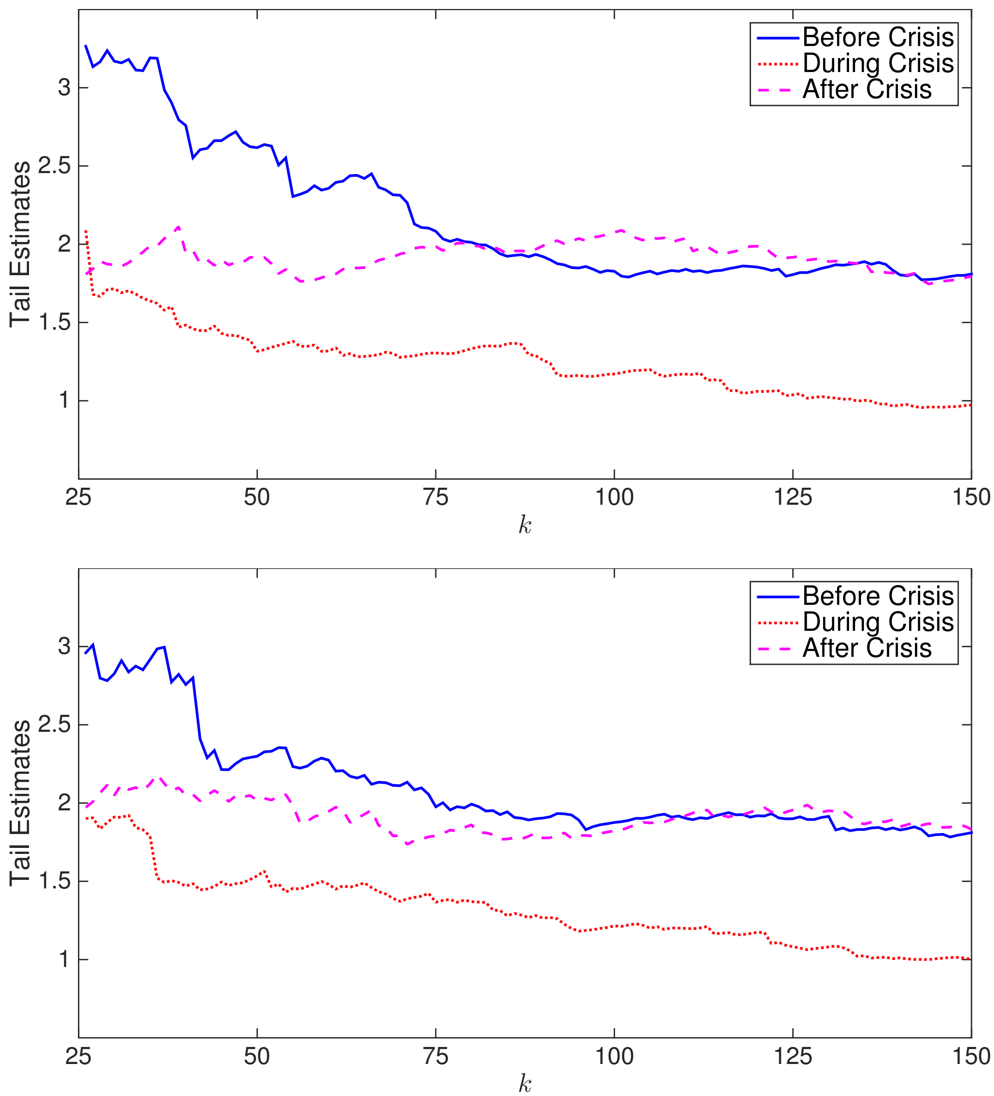


Fig. 3. **Tail index of bipower and threshold volatility measures over sub-periods.** The figures depict the Hill estimator of the tail index of the SPDR S&P 500 ETF (SPY). The full period is divided in three sub-periods: Before Crisis (June 15, 2004 through August 29, 2008), During Crisis (September 2, 2008 through May 29, 2009) and After Crisis (June 1, 2009 through June 13, 2014). The first panel depicts the tail index of the bipower volatility measure (open-to-close) for the three periods. The second panel depicts the tail index of the threshold volatility measure for the three periods.

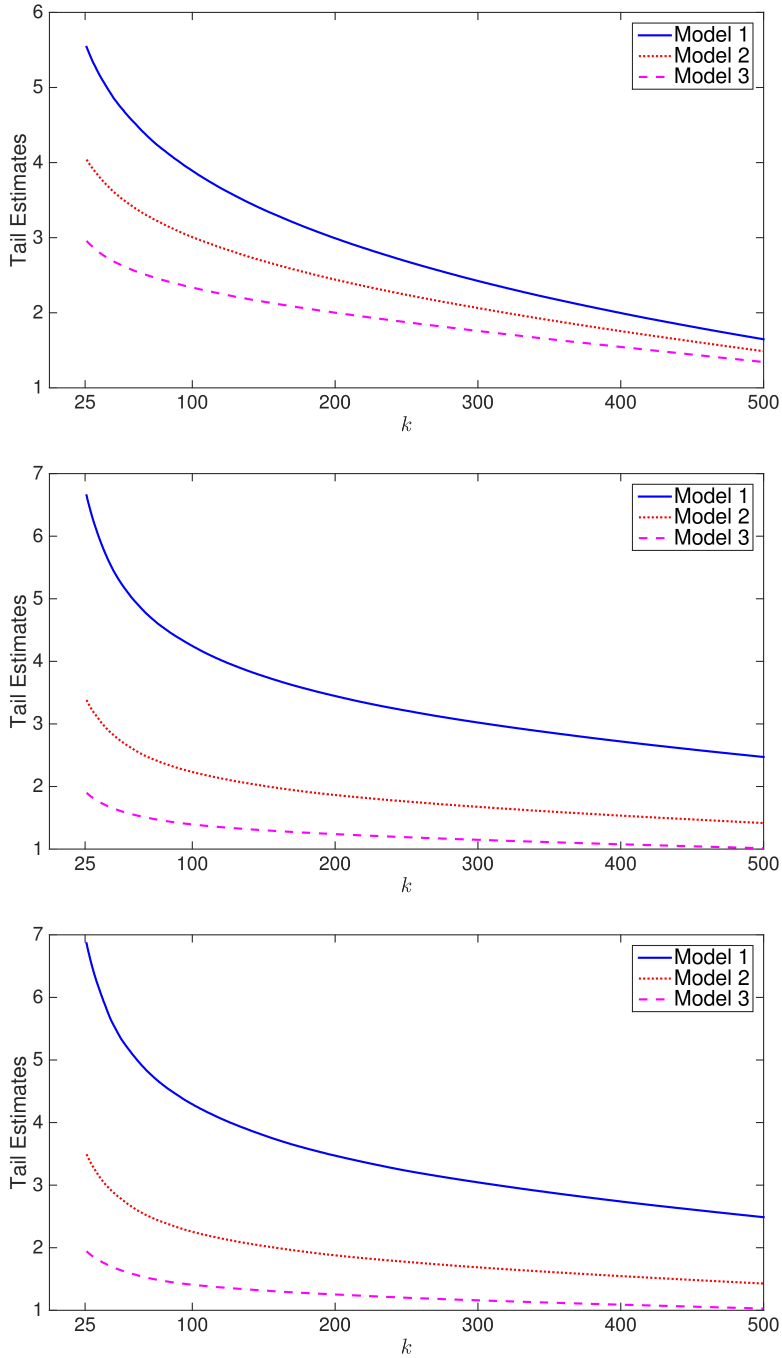


Fig. 4. **Tail index of returns, spot and integrated volatilities of the GARCH diffusion model.** The figures depict the average estimator of the tail index over 10,000 simulations of sample of 1,000 observations of the GARCH diffusion model. Three designs are considered: Model 1 corresponds to $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$, while Models 2 and 3 correspond respectively to $(0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$. The first panel depicts the tail index of daily returns; the second panel depicts the tail index of the daily spot volatility while the third one depicts the tail of the daily integrated volatility.

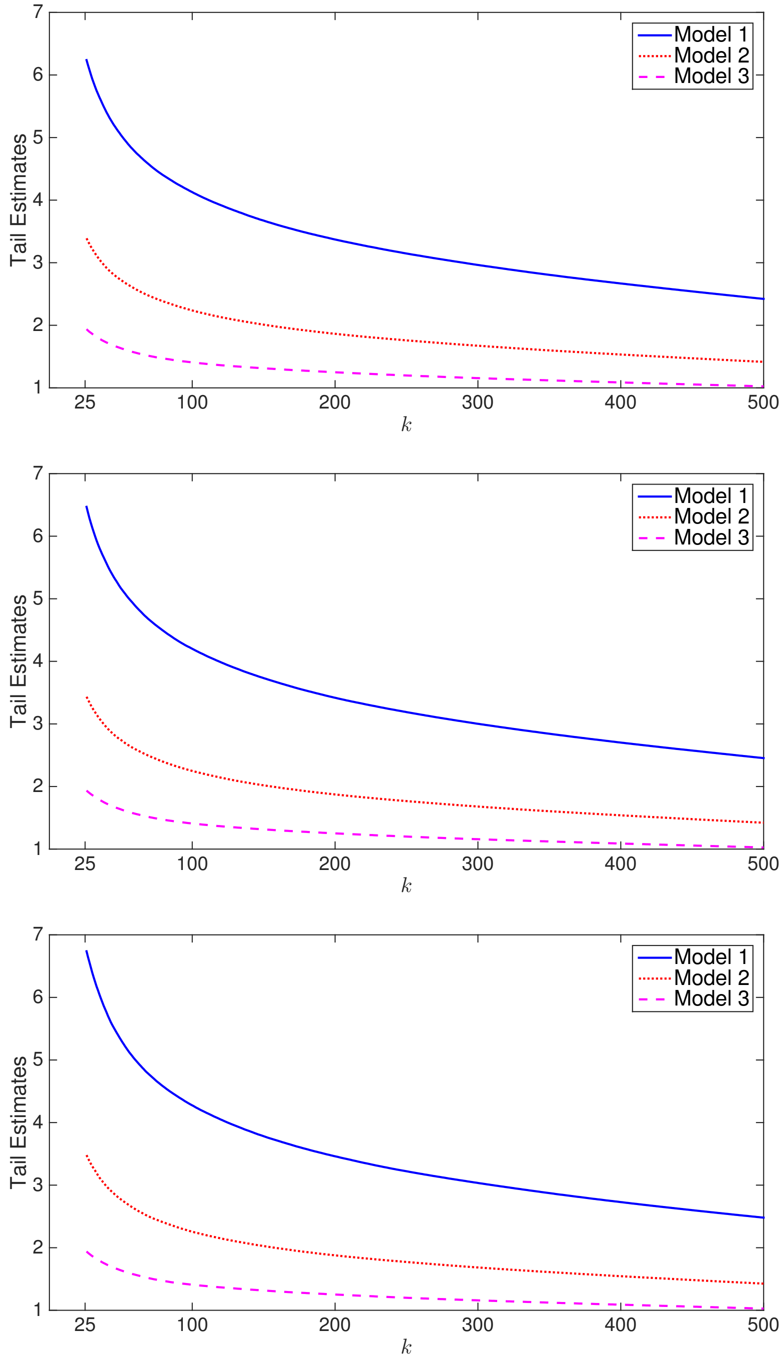


Fig. 5. **Tail index of realized volatility measures of the GARCH diffusion model.** The figures depict the average estimator of the tail index over 10,000 simulations of sample of 1,000 observations of the GARCH diffusion model. Three designs are considered: Model 1 corresponds to $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$, while Models 2 and 3 correspond respectively to $(0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$. The three panels depict the tail index of daily realized volatility with different frequencies: Panel 1 with 10 minute-returns RV; Panel 2 with 5 minute-returns RV; Panel 3 with 1 minute-returns RV.

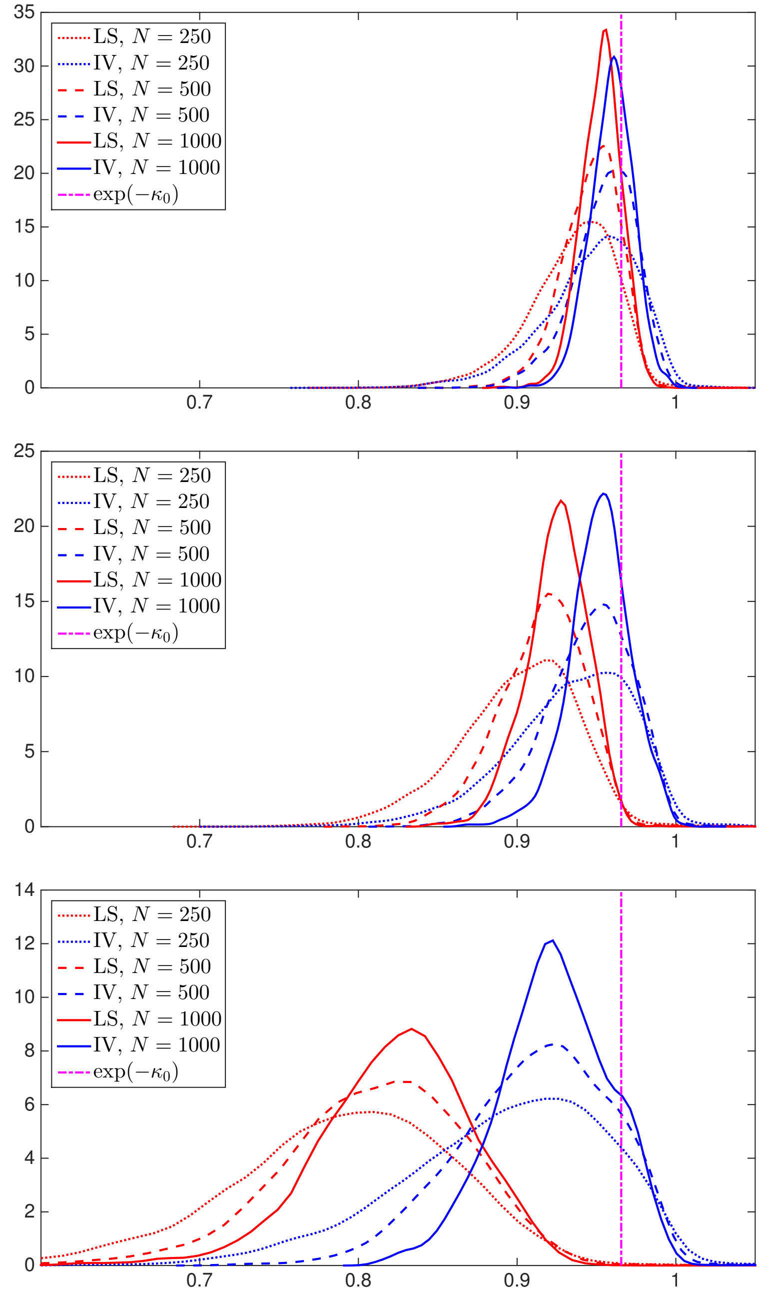


Fig. 6. **Autoregression estimation for spot volatility.** The figures depict the empirical distribution of the OLS and IV estimators of the autoregression of order one of daily spot volatility of the GARCH diffusion model. The figures are based on 10,000 simulations for three different sample sizes (250, 500 and 1,000). The instrument is the sign of the demeaned lagged value of the spot volatility. The first panel corresponds to Model 1 with $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$, while the second and their panels are for Models 2 and 3 that correspond respectively to $(0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$.

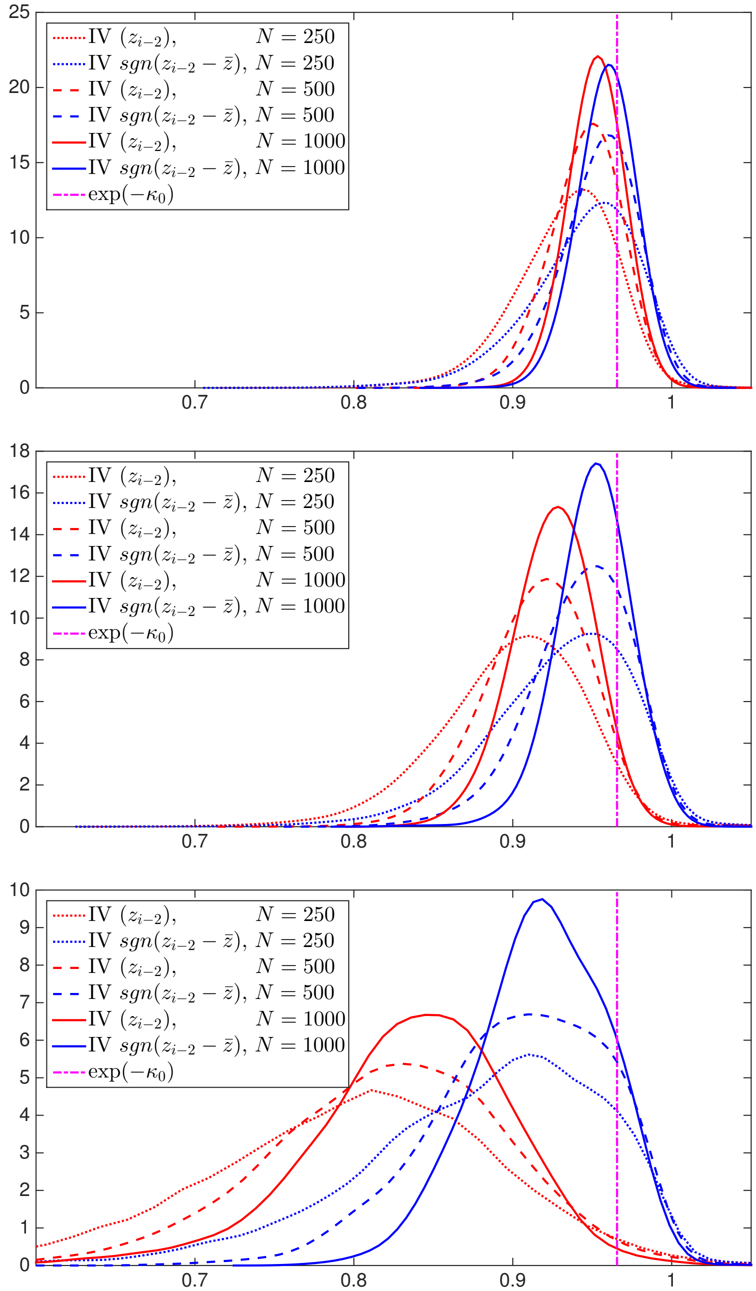


Fig. 7. **Multiperiod moment restriction for spot volatility.** The figures depict the empirical distribution of two IV estimators of the multiperiod-moment restrictions of the spot volatility of the GARCH diffusion model. The figures are based on 10,000 simulations for three different sample sizes (250, 500 and 1,000). The first instrument is the two lags of spot volatility while the second instrument is the sign of the demeaned value of the first instrument. The first panel corresponds to Model 1 with $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$, while the second and their panels are for Models 2 and 3 that correspond respectively to $(0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$.

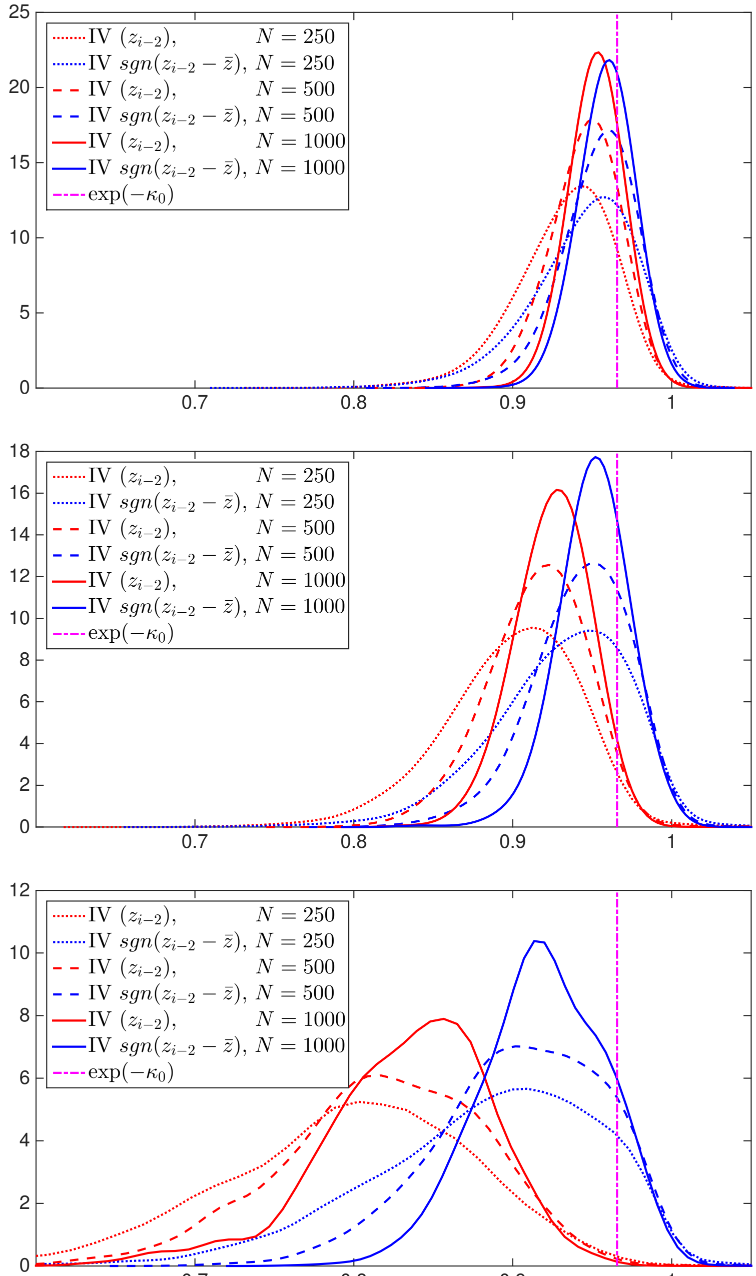


Fig. 8. **Multiperiod moment restriction for integrated volatility.** The figures depict the empirical distribution of two IV estimators of the multiperiod-moment restrictions of the integrated volatility of the GARCH diffusion model. The figures are based on 10,000 simulations for three different sample sizes (250, 500 and 1,000). The first instrument is the two lags of the integrated volatility while the second instrument is the sign of the demeaned value of the first instrument. The first panel corresponds to Model 1 with $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$, while the second and their panels are for Models 2 and 3 that correspond respectively to $(0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$.

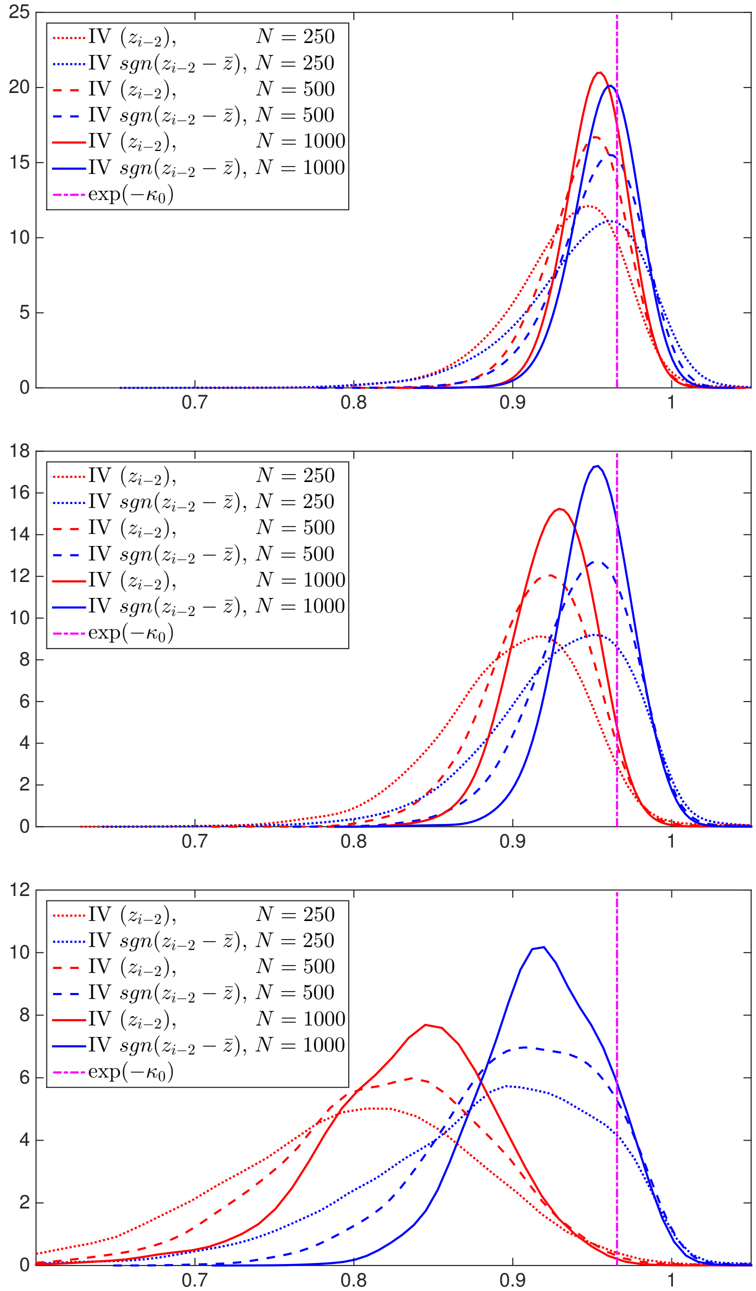


Fig. 9. **Multiperiod moment restriction for 10-minutes realized volatility.** The figures depict the empirical distribution of two IV estimators of the multiperiod-moment restrictions of the 10 minutes realized volatility of the GARCH diffusion model. The figures are based on 10,000 simulations for three different sample sizes (250, 500 and 1,000). The first instrument is the two lags of the 10 minutes realized volatility while the second instrument is the sign of the demeaned value of the first instrument. The first panel corresponds to Model 1 with $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$, while the second and their panels are for Models 2 and 3 that correspond respectively to $(0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$.

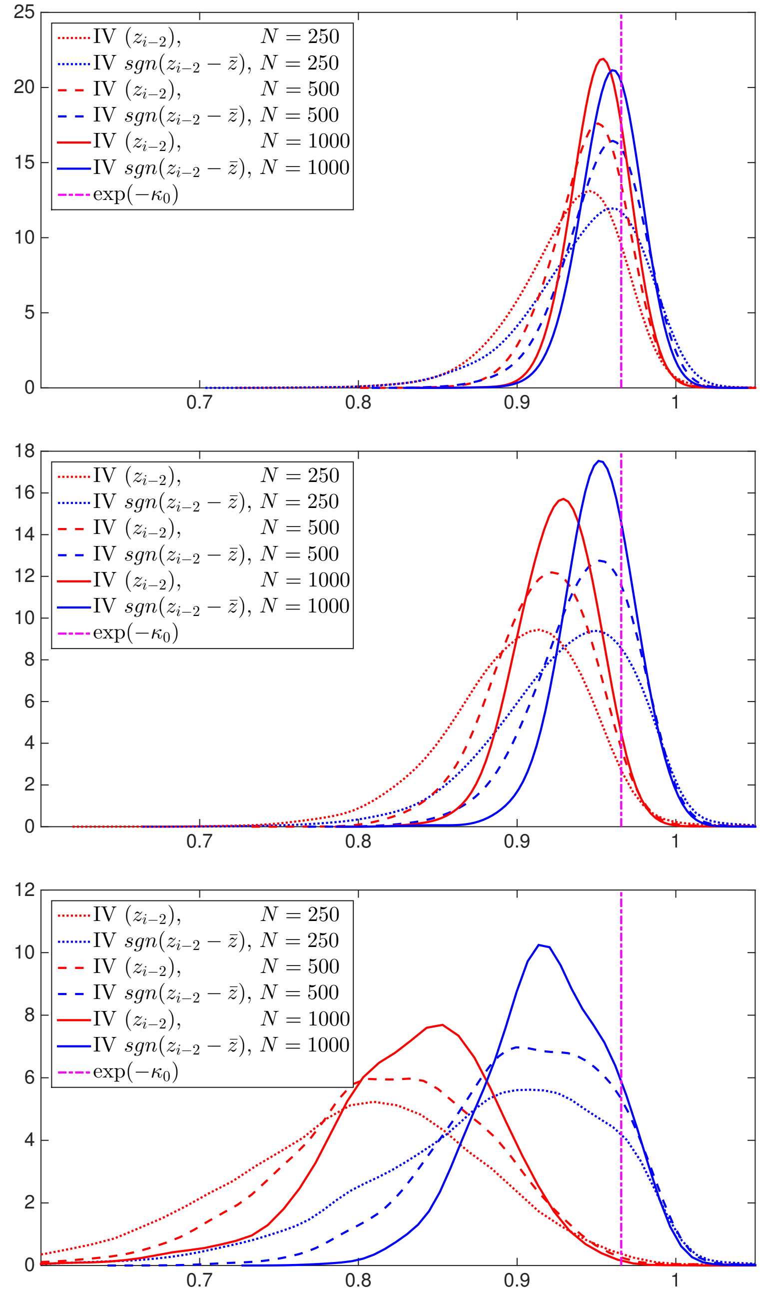


Fig. 10. **Multiperiod moment restriction for 5-minutes realized volatility.** The figures depict the empirical distribution of two IV estimators of the multiperiod-moment restrictions of the 5 minutes realized volatility of the GARCH diffusion model. The figures are based on 10,000 simulations for three different sample sizes (250, 500 and 1,000). The first instrument is the two lags of the 5 minutes realized volatility while the second instrument is the sign of the demeaned value of the first instrument. The first panel corresponds to Model 1 with $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$, while the second and their panels are for Models 2 and 3 that correspond respectively to $(0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$.

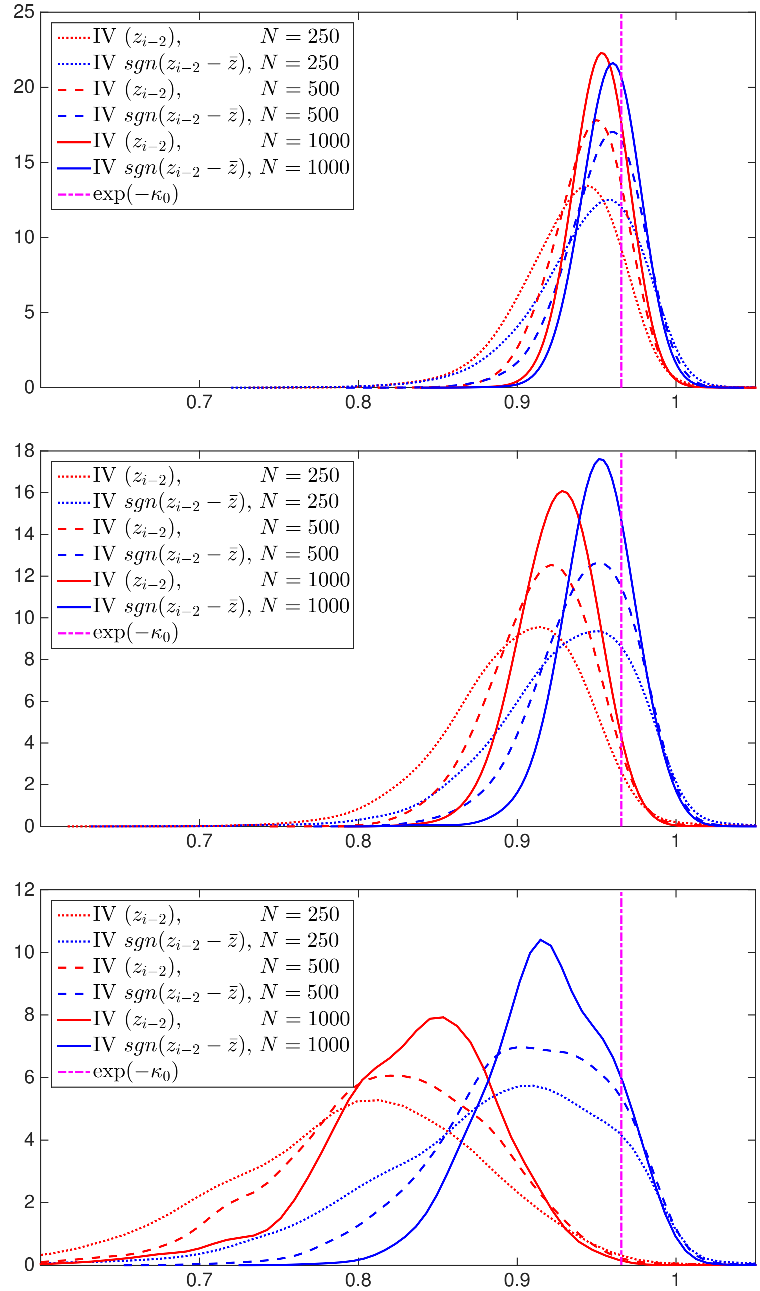


Fig. 11. **Multiperiod moment restriction for 1-minute realized volatility.** The figures depict the empirical distribution of two IV estimators of the multiperiod-moment restrictions of the 1 minutes realized volatility of the GARCH diffusion model. The figures are based on 10,000 simulations for three different sample sizes (250, 500 and 1,000). The first instrument is the two lags of the 1 minutes realized volatility while the second instrument is the sign of the demeaned value of the first instrument. The first panel corresponds to Model 1 with $(\kappa_0, \mu_0, \sigma_0^2) = (0.0350, 0.6360, 0.0207)$, while the second and their panels are for Models 2 and 3 that correspond respectively to $(0.0350, 0.6360, 0.0828)$ and $(0.0350, 0.6360, 0.3312)$.